

1

THE OPTIMAL DRAWINGS OF  $K_{5,n}$ 

2

CÉSAR HERNÁNDEZ-VÉLEZ, CAROLINA MEDINA, AND GELASIO SALAZAR

ABSTRACT. Zarankiewicz's Conjecture (ZC) states that the crossing number  $\text{cr}(K_{m,n})$  equals  $Z(m, n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ . Since Kleitman's verification of ZC for  $K_{5,n}$  (from which ZC for  $K_{6,n}$  easily follows), very little progress has been made around ZC; the most notable exceptions involve computer-aided results. With the aim of gaining a more profound understanding of this notoriously difficult conjecture, we investigate the *optimal* (that is, crossing-minimal) drawings of  $K_{5,n}$ . The widely known natural drawings of  $K_{m,n}$  (the so-called *Zarankiewicz drawings*) with  $Z(m, n)$  crossings contain *antipodal* vertices, that is, pairs of degree- $m$  vertices such that their induced drawing of  $K_{m,2}$  has no crossings. Antipodal vertices also play a major role in Kleitman's inductive proof that  $\text{cr}(K_{5,n}) = Z(5, n)$ . We explore in depth the role of antipodal vertices in optimal drawings of  $K_{5,n}$ , for  $n$  even. We prove that if  $n \equiv 2 \pmod{4}$ , then every optimal drawing of  $K_{5,n}$  has antipodal vertices. We also exhibit a two-parameter family of optimal drawings  $D_{r,s}$  of  $K_{5,4(r+s)}$  (for  $r, s \geq 0$ ), with no antipodal vertices, and show that if  $n \equiv 0 \pmod{4}$ , then every optimal drawing of  $K_{5,n}$  without antipodal vertices is (vertex rotation) isomorphic to  $D_{r,s}$  for some integers  $r, s$ . As a corollary, we show that if  $n$  is even, then every optimal drawing of  $K_{5,n}$  is the superimposition of Zarankiewicz drawings with a drawing isomorphic to  $D_{r,s}$  for some nonnegative integers  $r, s$ .

3

## 1. INTRODUCTION.

4

We recall that the *crossing number*  $\text{cr}(G)$  of a graph  $G$  is the minimum number of pairwise crossings of edges in a drawing of  $G$  in the plane. A drawing of a graph is *good* if no adjacent edges cross, and no two edges cross each other more than once. It is trivial to show that every *optimal* (that is, crossing-minimal) drawing of a graph is good.

8

One of the most tantalizingly open crossing number questions was raised by Turán in 1944: what is the crossing number  $\text{cr}(K_{m,n})$  of the complete bipartite graph  $K_{m,n}$ ? Zarankiewicz [8] described how to draw  $K_{m,n}$  with

11

---

*Date:* October 23, 2012.

*2010 Mathematics Subject Classification.* 05C10, 05C62, 68R10.

*Key words and phrases.* Crossing number, Turán's Brickyard Problem, Zarankiewicz Conjecture, optimal drawings, antipodal vertices.

The third author was supported by CONACYT grant 106432.

12 exactly  $Z(m, n)$  crossings, where

$$Z(m, n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

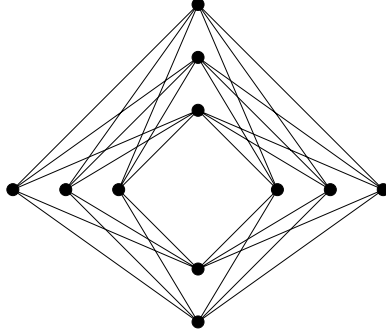


FIGURE 1. Drawing of  $K_{5,6}$  with  $Z(5, 6) = 24$  crossings.

13 Zarankiewicz's construction is shown in Figure 1 for the case  $m = 5, n = 6$ .  
 14 It is straightforward to generalize this drawing to a drawing of  $K_{m,n}$  with  
 15  $Z(m, n)$  crossings, for all positive integers  $m$  and  $n$ , and so  $\text{cr}(K_{m,n}) \leq$   
 16  $Z(m, n)$ . The drawings thus obtained are the *Zarankiewicz drawings* of  
 17  $K_{m,n}$ .

18 In [8], Zarankiewicz claimed to have proved that  $\text{cr}(K_{m,n}) = Z(m, n)$  for  
 19 all positive integers  $m, n$ . However, Kainen and Ringel independently found  
 20 a flaw in Zarankiewicz's argument (see [5]), and the statement  $\text{cr}(K_{m,n}) =$   
 21  $Z(m, n)$  has become known as *Zarankiewicz's Conjecture*.

22 Very little of substance is known about  $\text{cr}(K_{m,n})$ . An elegant argument us-  
 23 ing  $\text{cr}(K_{3,3}) = 1$  plus purely combinatorial arguments (namely, Turán's the-  
 24 orem on the maximum number of edges in a triangle-free graph) shows that  
 25  $\text{cr}(K_{3,n}) = Z(3, n)$ . An easy counting argument shows that  $\text{cr}(K_{2s-1,n}) =$   
 26  $Z(2s-1, n)$  (for any  $s \geq 1$ ) implies that  $\text{cr}(K_{2s,n}) = Z(2s, n)$ . Thus it fol-  
 27 lows that  $\text{cr}(K_{4,n}) = Z(4, n)$ . Kleitman [6] proved that  $\text{cr}(K_{5,n}) = Z(5, n)$ .  
 28 By our previous remark, this implies that  $\text{cr}(K_{6,n}) = Z(6, n)$ .

29 After Kleitman's theorem, most progress around Zarankiewicz's Conjec-  
 30 ture consists of computer-aided results. Woodall [7] verified Zarankiewicz's  
 31 Conjecture for  $K_{7,7}$  and  $K_{7,9}$ . De Klerk et al. [4] used semidefinite pro-  
 32 gramming techniques to show that  $\lim_{n \rightarrow \infty} \text{cr}(K_{7,n})/Z(7, n) \geq 0.968$ . Also  
 33 using semidefinite programming and deeper algebraic techniques, De Klerk  
 34 et al. [2] proved that  $\lim_{n \rightarrow \infty} \text{cr}(K_{9,n})/Z(9, n) \geq 0.966$ . In a related result,  
 35 De Klerk and Pasechnik [3] recently showed that the 2-page crossing number  
 36  $\nu_2(K_{7,n})$  of  $K_{7,n}$  satisfies  $\lim_{n \rightarrow \infty} \text{cr}(K_{7,n})/Z(7, n) = 1$ .

37 We finally mention that recently Christian et al. [1] proved that deciding  
 38 Zarankiewicz's Conjecture is a finite problem for each fixed  $m$ .

39 To give a brief description of our results, let us color the 5 degree- $n$  vertices  
 40 of  $K_{5,n}$  *black*, and color the  $n$  degree-5 vertices *white*. Two white vertices  
 41 are *antipodal* in a drawing  $D$  of  $K_{5,n}$  if the drawing of the  $K_{5,2}$  they induce  
 42 has no crossings. A drawing is *antipodal-free* if it has no antipodal vertices.

43 Antipodal pairs are evident in Zarankiewicz's drawings (moreover, the  
 44 set of white vertices can be decomposed into two classes, such that any two  
 45 white vertices in distinct classes are antipodal). Antipodal pairs are also  
 46 crucial in the inductive step of Kleitman's proof, which does not concern  
 47 itself with the different ways (if more than one) to achieve  $Z(5, n)$  crossings  
 48 with a drawing of  $K_{5,n}$ .

49 Given their preeminence in Zarankiewicz's Conjecture, we set out to in-  
 50 vestigate the role of antipodal pairs in the optimal drawings of  $K_{5,n}$ . Our  
 51 main result (Theorem 1) characterizes optimal drawings of  $K_{5,n}$ , for even  $n$ ,  
 52 as follows. First, if  $n \equiv 2 \pmod{4}$ , then all optimal drawings of  $K_{5,n}$  have  
 53 antipodal pairs. Second, if  $n \equiv 0 \pmod{4}$ , then every antipodal-free opti-  
 54 mal drawing of  $K_{5,n}$  is isomorphic (we review vertex rotation isomorphism  
 55 in Section 2) to a drawing in a two-parameter family  $D_{r,s}$  of drawings we  
 56 have fully characterized. As a consequence of these facts, we show (Theo-  
 57 rem 2) that if  $n$  is even, then every optimal drawing of  $K_{5,n}$  can be obtained  
 58 by starting with  $D_{r,s}$ , for some nonnegative (possibly zero) integers  $r$  and  $s$ ,  
 59 and then superimposing Zarankiewicz drawings.

60 The rest of this paper is organized as follows. In Section 2 we review the  
 61 concept of vertex rotation, which is central to the criterion to decide when  
 62 two drawings are isomorphic. In Section 3 we describe the two-parameter  
 63 family of optimal, antipodal-free drawings  $D_{r,s}$  (for integers  $r, s \geq 0$ ) of  
 64  $K_{5,4(r+s)}$ . In Section 4 we state our main results. Theorem 1 claims that (i)  
 65 if  $n \equiv 2 \pmod{4}$ , then every optimal drawing of  $K_{5,n}$  has antipodal vertices;  
 66 and that (ii) if  $n \equiv 0 \pmod{4}$ , then every antipodal-free optimal drawing of  
 67  $K_{5,n}$  is isomorphic to  $D_{r,s}$  for some integers  $r, s$  such that  $4(r+s) = n$ . In  
 68 Theorem 2 we state the decomposition of optimal drawings of  $K_{5,n}$ , along  
 69 the lines of the previous paragraph. The proof of Theorem 2 is also given  
 70 in this section; the rest of the paper is devoted to the proof of Theorem 1.  
 71 In Section 5 we introduce the concept of a *clean* drawing. Loosely speaking,  
 72 a drawing is clean if its white vertices can be naturally partitioned into  
 73 *bags*, so that vertices in the same bag have the same (crossing number wise)  
 74 properties. In Section 6 we introduce *keys*, which are labelled graphs that  
 75 capture the essential (crossing number wise) information of a clean drawing.  
 76 This abstraction (and the related concept of *core*) will prove to be extremely  
 77 useful for the proof of Theorem 1. In Section 7 we investigate which labelled  
 78 graphs can be the key of a relevant (clean, optimal, antipodal-free) drawing.  
 79 Cores are certain more manageable subgraphs of keys, that retain all the  
 80 (crossing number wise) useful information of a key. We devote Sections 8,  
 81 9, 10, and 11 to the task of completely characterizing which graphs can be

the core of an antipodal-free optimal drawing. The information in these sections is then put together in Section 12, where we show that the core of every optimal drawing is isomorphic either to the 4-cycle or to the graph  $\overline{C}_6$  obtained by adding to the 6-cycle a diametral edge. The proof of Theorem 1, given in Section 13, is an easy consequence of this full characterization of cores.

## 2. ROTATIONS AND ISOMORPHIC DRAWINGS.

To help comprehension, throughout this paper we color the 5 degree- $n$  vertices in  $K_{5,n}$  *black*, and the  $n$  degree-5 vertices *white*. We label the black vertices  $0, 1, 2, 3, 4$ . Unless otherwise stated, we label the white vertices  $a_0, a_1, \dots, a_{n-1}$ . We adopt the notation  $[n] := \{0, 1, \dots, n-1\}$ .

Given vertices  $a_i, a_j$  with  $i, j \in [n]$ , we let  $S(a_i)$  denote the *star* centered at  $a_i$ , that is, the subgraph (isomorphic to  $K_{5,1}$ ) induced by  $a_i$  and the vertices  $0, 1, 2, 3, 4$ . If  $D$  is a drawing of  $K_{5,n}$ , we let  $\text{cr}_D(a_i, a_j)$  denote the number of crossings in  $D$  that involve an edge of  $S(a_i)$  and an edge of  $S(a_j)$ , and we let  $\text{cr}_D(a_i) := \sum_{k \in [n], k \neq i} \text{cr}_D(a_i, a_k)$ . Formalizing the definition from Section 1,  $a_i$  and  $a_j$  are *antipodal* (in  $D$ ) if  $\text{cr}_D(a_i, a_j) = 0$ .

The *rotation*  $\text{rot}_D(a_i)$  of a white vertex  $a_i$  in a drawing  $D$  is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave  $a_i$ . We use the notation 01234 for permutations, and (01234) for cyclic permutations. For instance, the rotation  $\text{rot}_D(a_3)$  of the vertex  $a_3$  in the drawing  $D$  in Figure 2 is (02431): following a counterclockwise order, if we start with the edge leaving from  $a_3$  to 0, then we encounter the edge leaving to 2, then the edge leaving to 4, then the edge leaving to 3, and then the edge leaving to 1. We emphasize that a rotation is a cyclic permutation; that is, (02431), (24310), (43102), (31024), and (10243) denote (are) the same rotation. We let  $\Pi$  denote the set of all cyclic permutations of  $0, 1, 2, 3, 4$ . Clearly,  $|\Pi| = 5!/5 = 4! = 24$ . The *rotation*  $\text{rot}_D(i)$  of a black vertex  $i$  is defined analogously: for each  $i \in [5]$ ,  $\text{rot}_D(i)$  is a cyclic permutation of  $a_0, a_1, \dots, a_{n-1}$ .

The *rotation multiset*  $\text{Rot}_M(D)$  of  $D$  is the multiset (that is, repetitions are allowed) containing the  $n$  rotations  $\text{rot}_D(a_i)$ , for  $i = 0, 1, \dots, n-1$ . The *rotation set*  $\text{Rot}(D)$  of  $D$  is the underlying set (that is, no repetitions allowed) of  $\text{Rot}_M(D)$ . Thus, in the example of Figure 2,  $\text{Rot}_M(D) = [(04321), (04321), (01234), (02431)]$  (we use square brackets for multisets), and  $\text{Rot}(D) = \{(04321), (01234), (02431)\}$ .

Two multisets  $M, M'$  of rotations are *equivalent* (we write  $M \cong M'$ ) if one of them can be obtained from the other by a relabelling (formally, a self-bijection) of  $0, 1, 2, 3, 4$ . Two drawings  $D, D'$  of  $K_{5,n}$  are *isomorphic* if  $\text{Rot}_M(D) \cong \text{Rot}_M(D')$ . Loosely speaking, two drawings  $D, D'$  of  $K_{5,n}$  are isomorphic if  $0, 1, 2, 3, 4$  and  $a_0, a_1, \dots, a_{n-1}$  can be relabelled (say in  $D'$ ), if necessary, so that  $\text{rot}_D(a_i) = \text{rot}_{D'}(a_i)$  for every  $i \in [n]$ .

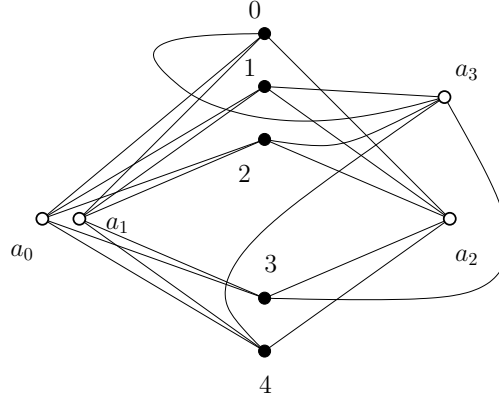


FIGURE 2. A drawing  $D$  of  $K_{5,4}$  with  $\text{rot}_D(a_0) = \text{rot}_D(a_1) = (04321)$ ,  $\text{rot}_D(a_2) = (01234)$ , and  $\text{rot}_D(a_3) = (02431)$ . Thus the pair  $a_0, a_2$  (as well as the pair  $a_1, a_3$ ) is antipodal.

Our ultimate interest lies in optimal drawings (of  $K_{5,n}$ ). It is not difficult to see (we will prove this later) that if  $D$  is an optimal drawing and  $a_i, a_j, a_k, a_\ell$  are vertices such that  $\text{rot}_D(a_i) = \text{rot}_D(a_j)$  and  $\text{rot}_D(a_k) = \text{rot}_D(a_\ell)$ , then  $\text{cr}_D(a_i, a_k) = \text{cr}_D(a_j, a_\ell)$ . Thus an optimal drawing of  $K_{5,n}$  is adequately described by choosing a representative vertex of each rotation, and giving the information of how many vertices there are for each rotation. This supports the pertinence of focusing on the rotations as the criteria for isomorphism.

### 3. AN ANTIPODAL-FREE DRAWING OF $K_{5,4(r+s)}$

In this section we describe an antipodal-free drawing  $D_{r,s}$  of  $K_{5,4(r+s)}$ , for each pair  $r, s$  of nonnegative integers.

The construction is based on the drawing  $D^*$  of  $K_{5,6}$  in Figure 3. As shown, the rotations in  $D^*$  of the white vertices are  $\text{rot}_{D^*}(a_0) = (01234)$ ,  $\text{rot}_{D^*}(a_1) = (04231)$ ,  $\text{rot}_{D^*}(a_2) = (01342)$ ,  $\text{rot}_{D^*}(a_3) = (04312)$ ,  $\text{rot}_{D^*}(a_4) = (01432)$ ,  $\text{rot}_{D^*}(a_5) = (02314)$ .

It is immediately checked that  $D^*$  is antipodal-free. Note that  $D^*$  itself is not optimal, as it has  $25 = Z(5, 6) + 1$  crossings.

Suppose first that both  $r$  and  $s$  are positive. To obtain  $D_{r,s}$ , we add  $4(r+s) - 6$  white vertices to  $D^*$ . Now  $r-1$  of these vertices are drawn very close to  $a_1$ , and  $r-1$  are drawn very close to  $a_2$ ;  $s-1$  vertices are drawn very close to  $a_4$ , and  $s-1$  are drawn very close to  $a_5$ ; finally,  $r+s-1$  vertices are drawn very close to  $a_0$ , and  $r+s-1$  are drawn very close to  $a_3$ . It is intuitively clear what is meant by having  $a_i$  drawn “very close” to  $a_j$ . Formally, we require that: (i)  $a_i$  and  $a_j$  have the same rotation; (ii)  $\text{cr}_{D_{r,s}}(a_i, a_j) = 4$ ; and (iii) for any other vertex  $a_k$ ,  $\text{cr}_{D_{r,s}}(a_i, a_k) = \text{cr}_{D_{r,s}}(a_j, a_k)$ . These properties are easily satisfied by having the added vertex  $a_i$  drawn sufficiently close to

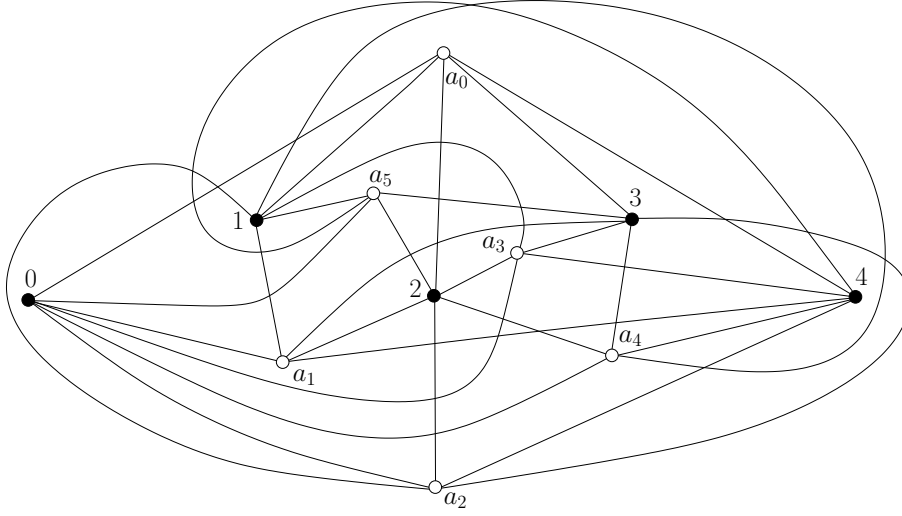


FIGURE 3. This antipodal-free drawing  $D^*$  of  $K_{5,6}$  is the base of the construction of the optimal antipodal-free drawing  $D_{r,s}$  of  $K_{5,4(r+s)}$  for all  $r, s$ . It is easily verified that  $\text{rot}_{D^*}(a_0) = (01234)$ ,  $\text{rot}_{D^*}(a_1) = (04231)$ ,  $\text{rot}_{D^*}(a_2) = (01342)$ ,  $\text{rot}_{D^*}(a_3) = (04312)$ ,  $\text{rot}_{D^*}(a_4) = (01432)$ ,  $\text{rot}_{D^*}(a_5) = (02314)$ .

150  $a_j$ , so that the edges incident with  $a_i$  follow very closely the edges incident  
151 with  $a_j$ .

152 If one of  $r$  or  $s$  is 0, then we make the obvious adjustments. That is, (i)  
153 if  $r = 0$ , then we remove  $a_1$  and  $a_2$ , and for each  $i = 0, 3, 4, 5$ , we draw  $s - 1$   
154 new vertices very close to  $a_i$ ; and (ii) if  $s = 0$ , then we remove  $a_4$  and  $a_5$ ,  
155 and for each  $i = 0, 1, 2, 3$ , we draw  $r - 1$  new vertices very close to  $a_i$ . (In  
156 the extreme case  $r = s = 0$ , we remove all the white vertices from  $D^*$ , and  
157 are left with an obviously optimal drawing of  $K_{5,0}$ ).

158 For each  $i = 0, 1, 2, 3, 4, 5$ , the *bag*  $[a_i]$  of  $a_i$  is the set that consists of the  
159 vertices drawn very close to  $a_i$ , plus  $a_i$  itself.

160 Note that each of  $[a_0]$  and  $[a_3]$  has  $r + s$  vertices, each of  $[a_1]$  and  $[a_2]$  has  
161  $r$  vertices, and each of  $[a_4]$  and  $[a_5]$  has  $s$  vertices.

162 An illustration of the construction for  $r = 2$  and  $s = 1$  is given in Figure 4,  
163 where the gray vertices are the ones added to  $D^*$ .

164 **Claim.** For every pair  $r, s$  of nonnegative integers,  $D_{r,s}$  is an antipodal-free  
165 optimal drawing of  $K_{5,4(r+s)}$ .

166 *Proof.* First we note that since  $D^*$  is antipodal-free, it follows immediately  
167 that  $D_{r,s}$  is also antipodal-free. Thus we only need to prove optimality.

168 An elementary calculation gives the number of crossings in  $D_{r,s}$ . For  
169 instance, take a vertex  $u$  in  $[a_0]$ . Now  $\text{cr}_{D_{r,s}}(u, v)$  equals (i) 4 if  $v \in [a_0], v \neq$   
170  $u$ ; (ii) 1 if  $v \in [a_1]$ ; (iii) 2 if  $v \in [a_2]$ ; (iv) 1 if  $v \in [a_3]$ ; (v) 1 if  $v \in [a_4]$ ; and (vi)  
171 2 if  $v \in [a_5]$ . Since  $|[a_0]| = r + s$ ,  $|[a_1]| = r$ ,  $|[a_2]| = r$ ,  $|[a_3]| = r + s$ ,  $|[a_4]| = s$ ,

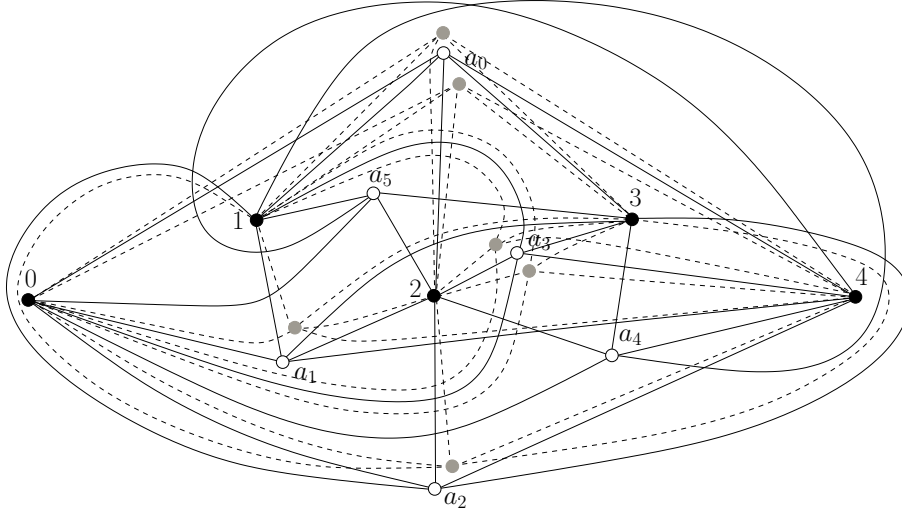


FIGURE 4. The antipodal-free drawing  $D_{2,1}$ . To obtain this optimal drawing of  $K_{5,12} = K_{5,4(2+1)}$ , we start with the drawing in Figure 3 and add two vertices very close to  $a_0$ , two vertices very close to  $a_3$ , one vertex very close to  $a_1$ , and one vertex very close to  $a_2$ . Since  $s - 1 = 0$ , no vertices are added very close to either  $a_4$  or  $a_5$ . The added vertices are colored gray in this drawing.

172 and  $|[a_5]| = s$ , it follows that  $\text{cr}_{D_{r,s}}(u) = 4(r+s-1) + r + 2r + (r+s) + s + 2s =$   
 173  $4(2r + 2s - 1)$ .

174 A totally analogous argument shows that, actually,  $\text{cr}_{D_{r,s}}(w) = 4(2r +$   
 175  $2s - 1)$  for *every* white vertex  $w$ . Since there are  $4(r + s)$  white vertices in  
 176 total, it follows that  $\text{cr}(D_{r,s}) = (1/2)(4(r + s))(4(2r + 2s - 1)) = (4(r +$   
 177  $s))(4(r + s) - 2) = Z(5, 4(r + s))$ .  $\square$

#### 178 4. MAIN RESULTS: THE OPTIMAL DRAWINGS OF $K_{5,n}$ , FOR $n$ EVEN.

179 We now state our main results.

180 **Theorem 1.** *Let  $n$  be a positive even integer.*

- 181 (1) *If  $n \equiv 2 \pmod{4}$ , then all optimal drawings of  $K_{5,n}$  have antipodal*  
 182 *vertices.*
- 183 (2) *If  $n \equiv 0 \pmod{4}$ , then every antipodal-free optimal drawing of  $K_{5,n}$*   
 184 *is isomorphic to  $D_{r,s}$  (described in Section 3) for some integers  $r, s$*   
 185 *such that  $4(r + s) = n$ .*

186 Before moving on to the proof of Theorem 1 (the rest of the paper is  
 187 devoted to this proof), we will show that it implies a decomposition of all  
 188 the optimal drawings of  $K_{5,n}$ , for  $n$  even.

189 In Section 1 we defined, somewhat informally, a Zarankiewicz drawing.  
 190 Let us now formally define these drawings using rotations (we focus on  
 191  $K_{5,n}$ , although the definition is obviously extended to  $K_{m,n}$  for any  $m$ ). For

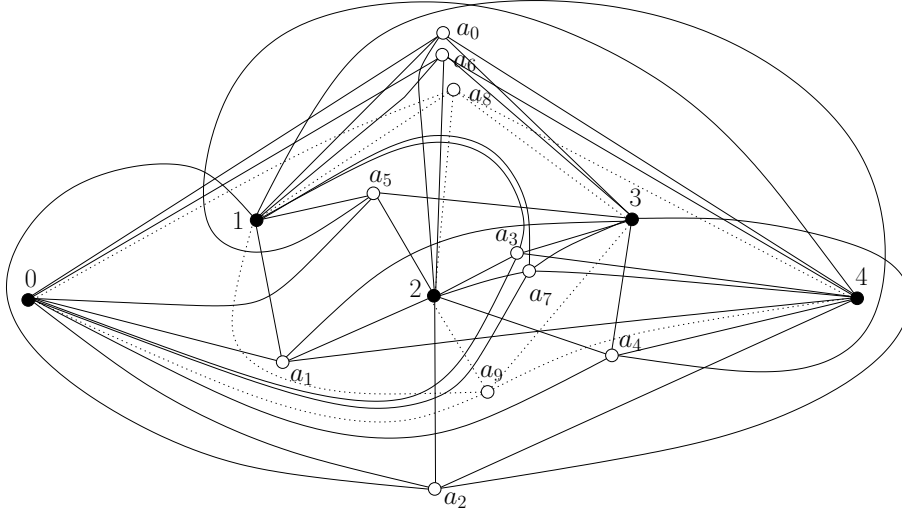


FIGURE 5. An optimal drawing of  $K_{5,10}$  that is neither a Zarankiewicz drawing nor the superimposition of Zarankiewicz drawings. As predicted by Theorem 2, this is the superimposition of a Zarankiewicz drawing (the  $K_{5,2}$  induced by  $a_8, a_9$  and the five black vertices) plus a drawing  $D_{r,s}$  (namely with  $r = s = 1$ ).

192 a nonnegative integer  $n$ , a drawing  $D$  of  $K_{5,n}$  is a *Zarankiewicz drawing* if  
 193 the white vertices can be partitioned into two sets, of sizes  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ ,  
 194 so that vertices in different sets are antipodal in  $D$ , and vertices  $a_i, a_j$  in the  
 195 same set satisfy  $\text{cr}_D(a_i, a_j) = 4$  (see Figure 1 for a Zarankiewicz drawing of  
 196  $K_{5,6}$ ). A quick calculation shows that every Zarankiewicz drawing of  $K_{5,n}$   
 197 is an optimal drawing.

198 **Theorem 2** (Decomposition of optimal drawings of  $K_{5,n}$ , for  $n$  even). *Let  $D$*   
 199 *be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Then the set of  $n$  white vertices*  
 200 *can be partitioned into two sets  $A, B$  (one of which may be empty), with  $|A| =$*   
 201  *$4t$  for some nonnegative integer  $t$ , such that: (i) the vertices in  $B$  can be*  
 202 *decomposed into  $|B|/2$  antipodal pairs; and (ii) the drawing of  $K_{5,4t}$  induced*  
 203 *by  $A$  is antipodal-free, and it is isomorphic to the drawing  $D_{r,s}$  described in*  
 204 *Section 3, for some integers  $r, s$  such that  $r + s = t$ . Equivalently, either*  
 205  *$D$  is the superimposition of Zarankiewicz drawings, or it can be obtained*  
 206 *by superimposing Zarankiewicz drawings to the drawing  $D_{r,s}$  described in*  
 207 *Section 3, for some integers  $r, s$  (see Figure 5).*

208 *Proof.* We proceed by induction on  $n$ . It is trivial to check that the two  
 209 white vertices of every optimal drawing of  $K_{5,2}$  are an antipodal pair, and  
 210 so the statement holds in the base case  $n = 2$ . For the inductive step, we  
 211 consider an even integer  $n$ , and assume that the statement is true for all  
 212  $k < n$ .



Let  $D$  be an optimal drawing of  $K_{5,n}$ . If  $D$  has no antipodal pairs, then the statement follows immediately from Theorem 1 (without even using the induction hypothesis). Thus we may assume that  $D$  has at least one antipodal pair  $a_i, a_j$ . It suffices to show that the drawing  $D'$  that results by removing  $a_i$  and  $a_j$  from  $D$  is an optimal drawing of  $K_{5,n-2}$ , as then the result follows by the induction hypothesis. Clearly  $\text{cr}(D) = \text{cr}(D') + \sum_{k \in [n] - \{i,j\}} (\text{cr}_D(a_i, a_k) + \text{cr}_D(a_j, a_k)) \geq \text{cr}(D') + (n-2)Z(5,3) = \text{cr}(D') + 4n-8$ . Thus  $\text{cr}(D') \leq \text{cr}(D) - 4n + 8 = Z(5,n) - 4n + 8$ . An elementary calculation shows that  $Z(5,n) - 4n + 8 = Z(5,n-2)$ , so we obtain  $\text{cr}(D') \leq Z(5,n-2)$ . Since  $\text{cr}(K_{5,n-2}) = Z(5,n-2)$ , it follows that  $\text{cr}(D') = Z(5,n-2)$ , that is,  $D'$  is an optimal drawing of  $K_{5,n-2}$ .  $\square$

## 5. CLEAN DRAWINGS.

A good drawing of  $K_{5,n}$  is *clean* if:

- (1) for all distinct white vertices  $a_i, a_j$  such that  $\text{rot}_D(a_i) = \text{rot}_D(a_j)$ , we have  $\text{cr}_D(a_i, a_j) = 4$ ;
- (2) for all distinct white vertices  $a_i, a_j, a_k, a_\ell$  such that  $\text{rot}_D(a_i) = \text{rot}_D(a_j)$  and  $\text{rot}_D(a_k) = \text{rot}_D(a_\ell)$ , we have  $\text{cr}_D(a_i, a_k) = \text{cr}_D(a_j, a_\ell)$ ; and
- (3) for any distinct white vertices  $a_i, a_k$ ,  $\text{cr}_D(a_i, a_k) \leq 4$ .

**Proposition 3.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ . Then there is an optimal drawing  $D'$ , isomorphic to  $D$ , that is clean.*

*Proof.* For each white vertex  $a_i$ , define  $d_i := \sum_{\{a_\ell \mid \text{rot}_D(a_\ell) \neq \text{rot}_D(a_i)\}} \text{cr}_D(a_i, a_\ell)$ . Let  $\pi \in \text{Rot}(D)$ . Take a white vertex  $a_i$  with  $\text{rot}_D(a_i) = \pi$ , such that for all  $j$  with  $\text{rot}_D(a_j) = \pi$  we have  $d_i \leq d_j$ . It is easy to see that we can move every vertex  $a_j$  with  $\text{rot}_D(a_j) = \pi$  very close to  $a_i$ , so that  $\text{cr}_D(a_i, a_k) = \text{cr}_D(a_j, a_k)$  for every white vertex  $a_k \notin \{a_i, a_j\}$ , and so that  $\text{cr}_D(a_i, a_j) = 4$ . If we perform this procedure for every rotation in  $\text{Rot}(D)$ , the result is an optimal drawing  $D'$ , isomorphic to  $D$ , that satisfies (1) and (2).

Now to prove that  $D'$  also satisfies (3) we suppose, by way of contradiction, that there exist  $a_i, a_k$  such that  $\text{cr}_D(a_i, a_k) > 4$ . Define  $d_i, d_k$  as in the previous paragraph. We may assume without loss of generality that  $d_i \leq d_k$ . Now let  $D''$  be the drawing that results from moving  $a_k$  very close to  $a_i$ , making it have the same rotation as  $a_i$ , and so that  $\text{cr}_{D''}(a_i, a_\ell) = \text{cr}_{D'}(a_i, a_\ell)$  for every  $\ell \notin \{i, k\}$ , and  $\text{cr}_{D''}(a_i, a_k) = 4$ . It is readily checked that  $D''$  has fewer crossings than  $D'$ , contradicting the optimality of  $D'$ .  $\square$

**Remark 4.** *We are interested in classifying optimal drawings up to isomorphism (Theorem 1). In view of Proposition 3, we may assume that all drawings of  $K_{5,n}$  under consideration are clean. We will work under this assumption for the rest of the paper.*

## 6. THE KEY OF A CLEAN DRAWING.

We now associate to every clean drawing of  $K_{5,n}$  an edge-labeled graph that (as we will see) captures all its relevant crossing number information.

Let  $D$  be a clean drawing of  $K_{5,n}$ . The *key*  $\Phi(D)$  of  $D$  is the (edge-labeled) complete graph whose vertices are the elements of  $\text{Rot}(D)$ , and where each edge is labeled according to the following rule: if  $\pi, \pi' \in \text{Rot}(D)$ , with  $\text{rot}_D(a_i) = \pi$  and  $\text{rot}_D(a_j) = \pi'$ , then the label of the edge joining  $\pi$  and  $\pi'$  is  $\text{cr}_D(a_i, a_j)$ . It follows from the cleanness of  $D$  that  $\text{cr}_D(a_i, a_j)$  does not depend on the choice of  $a_i$  and  $a_j$ , and so  $\Phi(D)$  is well-defined for every clean drawing  $D$ . Moreover, it also follows that every edge label in  $\Phi(D)$  is in  $\{0, 1, 2, 3, 4\}$ . The *core* of  $D$  is the subgraph  $\Phi^1(D)$  of  $\Phi(D)$  that consists of all the vertices of  $\Phi(D)$  and the edges of  $\Phi(D)$  with label 1. In Figure 6 we give a (clean and optimal) drawing  $D$  of  $K_{5,3}$ , and illustrate its key and its core.

Our main interest is in antipodal-free drawings, that is, those drawings in which every edge label in  $\Phi(D)$  is in  $\{1, 2, 3, 4\}$ . A key is *0-free* (respectively, *4-free*) if none of its edges has 0 (respectively, 4) as a label. A key is  $\{0, 4\}$ -free if it is both 0- and 4-free.

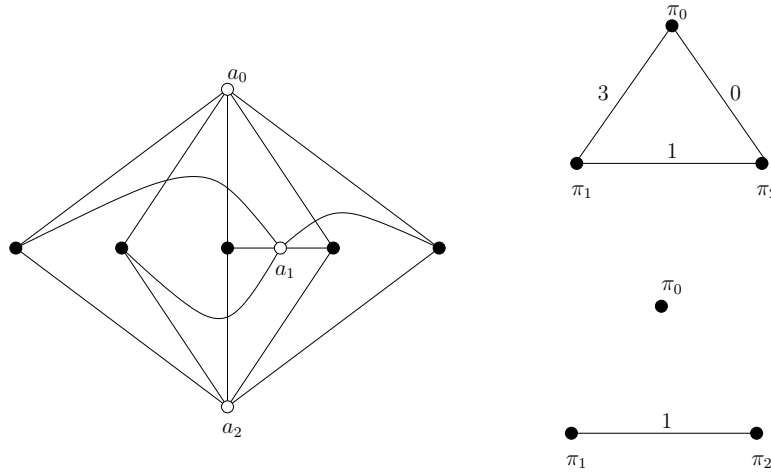


FIGURE 6. A drawing  $D$  of  $K_{5,3}$ . By letting  $\text{rot}_D(a_0) = \pi_0$ ,  $\text{rot}_D(a_1) = \pi_1$ , and  $\text{rot}_D(a_2) = \pi_2$ , we obtain the key  $\Phi(D)$  (right, above) and the core  $\Phi^1(D)$  (right, below) of  $D$ .

The main step in our strategy to understand optimal drawings is to characterize which labelled graphs are the key of some optimal drawing. To this end, we introduce a system of linear equations associated to each key, as follows.

**Definition 5** (The system of linear equations of a key). *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Let the vertices of  $\Phi(D)$  (that is, the elements of  $\text{Rot}(D)$ ) be labelled  $\pi_0, \pi_1, \dots, \pi_{m-1}$ , and let  $\lambda_{ij}$  denote the label of the edge  $\pi_i\pi_j$ , for all  $i \neq j$ . For each  $i \in [m]$ , the linear equation  $E(\pi_i, \Phi(D))$  for  $\pi_i$  in  $\Phi(D)$  is the linear equation on the variables  $t_0, t_1, \dots, t_{m-1}$  given*

278 by

$$E(\pi_i, \Phi(D)) : 2t_i + \sum_{j \in [m], j \neq i} (\lambda_{ij} - 2)t_j = 0.$$

279 The set  $\{E(\pi_i, \Phi(D))\}_{i \in [m]}$  is the system of linear equations associated  
 280 to  $\Phi(D)$ , and is denoted  $\mathcal{L}(\Phi(D))$ .

281 The characterization of when a labelled graph is the key of an optimal  
 282 drawing is mainly based on the following crucial fact.

283 **Proposition 6.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Then*  
 284 *the system of linear equations  $\mathcal{L}(\Phi(D))$  associated to  $\Phi(D)$  has a positive*  
 285 *integral solution  $(t_0, t_1, \dots, t_{m-1})$  such that  $t_0 + t_1 + \dots + t_{m-1} = n$ .*

286 *Proof.* First we show that if  $D$  is an optimal drawing of  $K_{5,n}$  with  $n$  even,  
 287 then for every  $i = 0, 1, \dots, n-1$ , we have  $\text{cr}_D(a_i) = 2n - 4$ . To this end,  
 288 suppose that  $\text{cr}_D(a_i) > 2n - 4$  for some  $i$ . Since  $D$  is optimal,  $\text{cr}(D) =$   
 289  $Z(5, n) = n(n-2)$ , and so the drawing  $D'$  of  $K_{5,n-1}$  that results by removing  
 290  $a_i$  from  $D$  has fewer than  $n(n-2) - (2n-4) = n^2 - 4n + 4 = (n-2)^2 =$   
 291  $Z(5, n-1)$  crossings, contradicting that  $\text{cr}(K_{5,n-1}) = Z(5, n-1)$ . Thus  
 292  $\text{cr}_D(a_i) \leq 2n - 4$  for every  $i$ . Now suppose that  $\text{cr}_D(a_i) < 2n - 4$  for  
 293 some  $i$ . Then  $\text{cr}(D) = (1/2) \sum_{j \in [n]} \text{cr}_D(a_j) < (1/2)(2n-4)n = n(n-2)$ ,  
 294 contradicting that  $\text{cr}(K_{5,n}) = Z(5, n) = n(n-2)$ . Thus for every  $i \in [n]$  we  
 295 have  $\text{cr}_D(a_i) = 2n - 4$ , as claimed.

296 Now let  $\pi_0, \pi_1, \dots, \pi_{m-1}$  be the elements of  $\text{Rot}(D)$  (that is, the vertices of  
 297  $\Phi(D)$ ), and for each  $i, j \in [m], i \neq j$ , let  $\lambda_{ij}$  denote the label of the edge  $\pi_i \pi_j$   
 298 in  $\Phi(D)$ . For each  $i \in [m]$ , let  $t_i$  be the number of vertices with rotation  $\pi_i$   
 299 in  $D$ . Then (using that  $D$  is clean) for every  $i \in [m]$  and every white vertex  
 300  $a_k$  with  $\text{rot}_D(a_k) = \pi_i$  we have  $\text{cr}_D(a_k) = 4(t_i - 1) + \sum_{j \in [m], j \neq i} \lambda_{ij} t_j$ . Now  
 301 from the previous paragraph for each  $a_k$  we have  $\text{cr}_D(a_k) = 2n - 4$ . Using  
 302 that  $n = \sum_{j \in [m]} t_j$ , we obtain  $4(t_i - 1) + \sum_{j \in [m], j \neq i} \lambda_{ij} t_j = 2 \sum_{j \in [m]} t_j -$   
 303  $4$ . Equivalently,  $2t_i + \sum_{j \in [m], j \neq i} (\lambda_{ij} - 2)t_j = 0$ , for every  $i \in [m]$ . Thus  
 304  $(t_0, t_1, \dots, t_{m-1})$  is a positive integral solution of  $\mathcal{L}(\Phi(D))$ .  $\square$

## 305 7. PROPERTIES OF THE KEY OF A CLEAN DRAWING.

306 We start with an easy, yet crucial, observation.

307 **Proposition 7.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ . Then, for any three*  
 308 *distinct white vertices  $a_i, a_j, a_k$ ,  $\text{cr}_D(a_i, a_j) + \text{cr}_D(a_j, a_k) + \text{cr}_D(a_i, a_k)$  is an*  
 309 *even number greater than or equal to 4.*

310 *Proof.* This follows since  $\text{cr}(K_{5,3}) = Z(5, 3) = 4$  and (see for instance [6])  
 311 every good drawing of  $K_{5,3}$  has an even number of crossings.  $\square$

312 The following is an equivalent form of this statement, in the setting of  
 313 keys.

**Proposition 8.** *Let  $D$  be a clean drawing of  $K_{5,n}$ , and let  $\pi_0, \pi_1, \pi_2$  be vertices of  $\Phi(D)$ . Let  $\lambda_{ij}$  be the label of the edge  $\pi_i\pi_j$ , for  $i, j \in \{0, 1, 2\}, i \neq j$ . Then  $\lambda_{01} + \lambda_{12} + \lambda_{02}$  is an even number greater than or equal to 4.  $\square$*

Let  $\gamma, \kappa$  be cyclic permutations on the same set of symbols. A *route* from  $\gamma$  to  $\kappa$  is a set of distinct transpositions, which may be ordered into some sequence such that the successive application of (all) the transpositions in this sequence takes  $\gamma$  to  $\kappa$ . For instance, if  $\gamma = (abcd)$  and  $\kappa = (acdb)$ , then  $\{(bd), (bc)\}$  is a route from  $\gamma$  to  $\kappa$ : if we apply first  $(bc)$  to  $\gamma$ , and then  $(bd)$  to the resulting cyclic permutation, we obtain  $\kappa$ .

The *size*  $|P|$  of a route  $P$  is its number of transpositions. An *antiroute* from  $\gamma$  to  $\kappa$  is a route from  $\gamma$  to the reverse cyclic permutation  $\bar{\kappa}$  of  $\kappa$ . Note that if  $P$  is a route (respectively, antiroute) from  $\gamma$  to  $\kappa$ , then  $P$  is also a route (respectively, antiroute) from  $\kappa$  to  $\gamma$ . The *antidistance* between two cyclic permutations is the smallest size of an antiroute between them.

The following is an easy consequence of (the proof of) Theorem 5 in [7].

**Lemma 9.** *Let  $D$  be a good drawing of  $K_{5,2}$ , with white vertices  $a_0, a_1$ . Then there is an antiroute from  $\text{rot}_D(a_0)$  to  $\text{rot}_D(a_1)$  of size  $\text{cr}_D(a_0, a_1)$ .  $\square$*

The following statement is implicitly proved in the discussion after the proof of [7, Theorem 5].

**Lemma 10.** *Let  $D$  be a clean drawing of  $K_{5,r}$  with white vertices  $a_0, a_1, \dots, a_{r-1}$ , and let  $\pi_i := \text{rot}_D(a_i)$ . Suppose that  $\pi_i \neq \pi_j$  whenever  $i \neq j$ , and for all  $i \neq j$  let  $\lambda_{ij} := \text{cr}_D(a_i, a_j)$ . For  $k = 0, 1, 2, 3, 4$ , let  $\gamma_k := \text{rot}_D(k)$ . Then there exist:*

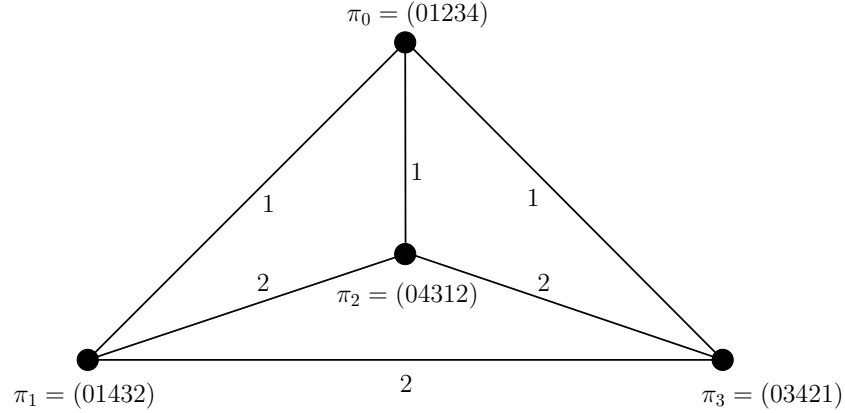
- (1) *for all  $i, j \in [r]$  with  $i \neq j$ , an antiroute  $P_{ij}$  from  $\pi_i$  to  $\pi_j$  of size  $\lambda_{ij}$ ;*
  - (2) *for all  $k, \ell \in [5]$  with  $k \neq \ell$ , an antiroute  $Q_{k\ell}$  from  $\gamma_k$  to  $\gamma_\ell$ ;*
- such that the transposition  $(a_i a_j)$  is in  $Q_{k\ell}$  if and only if the transposition  $(k \ell)$  is in  $P_{ij}$ .  $\square$*

We now use these powerful statements to prove that certain graphs cannot be the subgraphs of the key of a clean drawing.

**Proposition 11.** *The graph in Figure 7 is not the key of any clean drawing of  $K_{5,n}$ .*

*Proof.* Suppose by way of contradiction that the graph in Figure 7 is the key of some clean drawing of  $K_{5,n}$ . This implies in particular that there exists a drawing  $D$  of  $K_{5,4}$  with white vertices  $a_0, a_1, a_2, a_3$  such that  $\text{rot}_D(a_i) = \pi_i$  for  $i = 0, 1, 2, 3$ , with  $\pi_0 = (01234), \pi_1 = (01432), \pi_2 = (04312)$ , and  $\pi_3 = (03421)$ , and  $\text{cr}_D(a_0, a_1) = \text{cr}_D(a_0, a_2) = \text{cr}_D(a_0, a_3) = 1$ , and  $\text{cr}_D(a_1, a_2) = \text{cr}_D(a_1, a_3) = \text{cr}_D(a_2, a_3) = 2$ .

The required contradiction is obtained by showing that there do not exist rotations  $\text{rot}_D(0), \text{rot}_D(1), \text{rot}_D(2), \text{rot}_D(3), \text{rot}_D(4)$ , and antiroutes  $P_{ij}, Q_{k\ell}$  that satisfy Lemma 10 (with the given values of  $\text{cr}_D(a_i, a_j)$  for  $i, j \in \{0, 1, 2, 3\}, i \neq j$ ). We start by determining the possible antiroutes  $P_{ij}$  (these depend

FIGURE 7. This cannot be the key of a clean drawing of  $K_{5,n}$ .

only on the information we already have). Then we investigate the possible antiroutes  $Q_{kl}$  consistent with each choice of the antiroutes  $P_{ij}$ , and prove that, in all cases, every possible choice of  $\text{rot}_D(0), \text{rot}_D(1), \text{rot}_D(2), \text{rot}_D(3)$  and  $\text{rot}_D(4)$  leads to an inconsistency.

The following facts are easily verified: (i) the only antiroute from  $\pi_0$  to  $\pi_1$  of size 1 is  $\{(01)\}$ ; (ii) the only antiroute from  $\pi_0$  to  $\pi_2$  of size 1 is  $\{(12)\}$ ; (iii) the only antiroute from  $\pi_0$  to  $\pi_3$  of size 1 is  $\{(34)\}$ ; (iv) the only antiroute of size 2 from  $\pi_1$  to  $\pi_2$  is  $\{(02), (34)\}$ ; (v) there are two distinct antiroutes of size 2 from  $\pi_2$  to  $\pi_3$ , namely  $\{(01), (02)\}$  and  $\{(03), (04)\}$ ; and (vi) there are two distinct antiroutes of size 2 from  $\pi_1$  to  $\pi_3$ , namely  $\{(02), (12)\}$  and  $\{(23), (24)\}$ .

Now for  $i, j \in \{0, 1, 2, 3\}, i \neq j$ , let  $P_{ij}$  be the antiroute guaranteed by Lemma 10. By the previous observations it follows that necessarily  $P_{01} = \{(01)\}$ ,  $P_{02} = \{(12)\}$ ,  $P_{03} = \{(34)\}$ , and  $P_{12} = \{(02), (34)\}$ . Also by the previous observations there are two choices for  $P_{23}$ , namely  $\{(01), (02)\}$  and  $\{(03), (04)\}$ ; and there are two choices for  $P_{13}$ , namely  $\{(02), (12)\}$  and  $\{(23), (24)\}$ .

Thus  $P_{01}, P_{02}, P_{03}, P_{12}$  are all determined:

$$P_{01} = \{(01)\}, P_{02} = \{(12)\}, P_{03} = \{(34)\}, P_{12} = \{(02), (34)\},$$

and there are four possible combinations of  $P_{13}$  and  $P_{23}$ :

(a)  $P_{23} = \{(01), (02)\}$  and  $P_{13} = \{(02), (12)\}$ .

In this case, by Lemma 10, we have  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{02} = \{(a_1a_2), (a_2a_3), (a_1a_3)\}$ ,  $Q_{03} = \emptyset$ ,  $Q_{04} = \emptyset$ ,  $Q_{12} = \{(a_0a_2), (a_1a_3)\}$ ,  $Q_{13} = \emptyset$ ,  $Q_{14} = \emptyset$ ,  $Q_{23} = \emptyset$ ,  $Q_{24} = \emptyset$ , and  $Q_{34} = \{(a_0a_3), (a_1a_2)\}$ .

(b)  $P_{23} = \{(01), (02)\}$  and  $P_{13} = \{(23), (24)\}$ .

379 In this case, by Lemma 10, we have  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{02} =$   
 380  $\{(a_1a_2), (a_2a_3)\}$ ,  $Q_{03} = \emptyset$ ,  $Q_{04} = \emptyset$ ,  $Q_{12} = \{(a_0a_2)\}$ ,  $Q_{13} = \emptyset$ ,  $Q_{14} =$   
 381  $\emptyset$ ,  $Q_{23} = \{(a_1a_3)\}$ ,  $Q_{24} = \{(a_1a_3)\}$ , and  $Q_{34} = \{(a_0a_3), (a_1a_2)\}$ .

382 (c)  $P_{23} = \{(03), (04)\}$  and  $P_{13} = \{(02), (12)\}$ .

383 In this case, by Lemma 10, we have  $Q_{01} = \{(a_0a_1)\}$ ,  $Q_{02} = \{(a_1a_2),$   
 384  $(a_1a_3)\}$ ,  $Q_{03} = \{(a_2a_3)\}$ ,  $Q_{04} = \{(a_2a_3)\}$ ,  $Q_{12} = \{(a_0a_2), (a_1a_3)\}$ ,  
 385  $Q_{13} = \emptyset$ ,  $Q_{14} = \emptyset$ ,  $Q_{23} = \emptyset$ ,  $Q_{24} = \emptyset$ , and  $Q_{34} = \{(a_0a_3), (a_1a_2)\}$ .

386 (d)  $P_{23} = \{(03), (04)\}$  and  $P_{13} = \{(23), (24)\}$ .

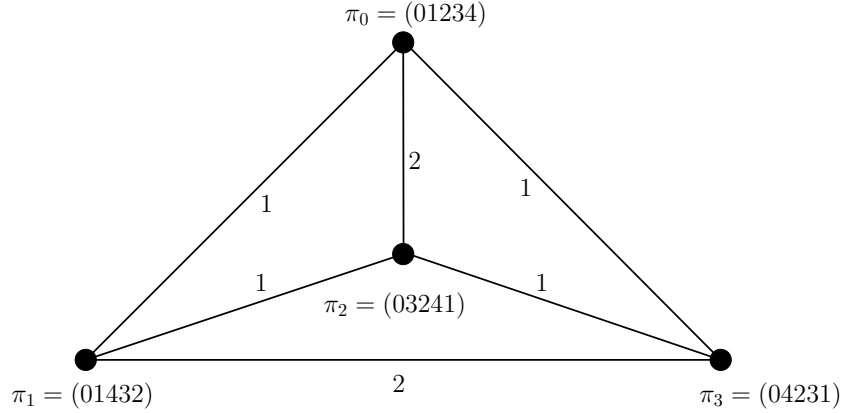
387 In this case, by Lemma 10, we have  $Q_{01} = \{(a_0a_1)\}$ ,  $Q_{02} = \{(a_1a_2)\}$ ,  
 388  $Q_{03} = \{(a_2a_3)\}$ ,  $Q_{04} = \{(a_2a_3)\}$ ,  $Q_{12} = \{(a_0a_2)\}$ ,  $Q_{13} = \emptyset$ ,  $Q_{14} =$   
 389  $\emptyset$ ,  $Q_{23} = \{(a_1a_3)\}$ ,  $Q_{24} = \{(a_1a_3)\}$ , and  $Q_{34} = \{(a_0a_3), (a_1a_2)\}$ .

390 We only analyze (that is, derive a contradiction from) (a). The cases (b),  
 391 (c), and (d) are handled in a totally analogous manner.

392 Since  $Q_{13} = Q_{14} = \emptyset$ , it follows that  $\text{rot}_D(3)$  and  $\text{rot}_D(4)$  are both equal  
 393 to the reverse of  $\text{rot}_D(1)$ ; in particular,  $\text{rot}_D(3) = \text{rot}_D(4)$ . Since  $Q_{01} =$   
 394  $\{(a_0a_1), (a_2a_3)\}$  and  $Q_{12} = \{(a_0a_2), (a_1a_3)\}$ , it follows that in  $\text{rot}_D(1)$ : (i)  
 395  $a_0$  and  $a_1$  must be adjacent; (ii)  $a_2$  and  $a_3$  must be adjacent; (iii)  $a_0$  and  
 396  $a_2$  must be adjacent; and (iv)  $a_1$  and  $a_3$  must be adjacent. It follows imme-  
 397 diately that  $\text{rot}_D(1)$  is either  $(a_0a_2a_3a_1)$  or  $(a_0a_1a_3a_2)$ . Since  $\text{rot}_D(3)$  and  
 398  $\text{rot}_D(4)$  are both the reverse of  $\text{rot}_D(1)$ , then each of  $\text{rot}_D(3)$  and  $\text{rot}_D(4)$   
 399 is either  $(a_0a_1a_3a_2)$  or  $(a_0a_2a_3a_1)$ . However, since  $Q_{34} = \{(a_0a_3), (a_1a_2)\}$ ,  
 400 then one must reach the reverse of  $\text{rot}_D(4)$  from  $\text{rot}_D(3)$  by applying the  
 401 transpositions  $(a_0a_3)$  and  $(a_1a_2)$  (in some order). Since neither of these  
 402 transpositions may be applied to  $(a_0a_1a_3a_2)$  or  $(a_0a_2a_3a_1)$ , we obtain the  
 403 required contradiction.  $\square$

404 **Proposition 12.** *The graph in Figure 8 is not the key of any clean drawing*  
 405 *of  $K_{5,n}$ .*

406 *Proof.* Suppose by way of contradiction that the graph in Figure 8 is the  
 407 key of some clean drawing of  $K_{5,n}$ . Thus there exists a drawing  $D$  of  $K_{5,4}$   
 408 with white vertices  $a_0, a_1, a_2, a_3$  such that  $\text{rot}_D(a_i) = \pi_i$  for  $i = 0, 1, 2, 3$ ,  
 409 with  $\pi_0 = (01234)$ ,  $\pi_1 = (01432)$ ,  $\pi_2 = (03241)$ , and  $\pi_3 = (04231)$ , and  
 410  $\text{cr}_D(a_0, a_1) = \text{cr}_D(a_1, a_2) = \text{cr}_D(a_2, a_3) = \text{cr}_D(a_0, a_3) = 1$ , and  $\text{cr}_D(a_0, a_2) =$   
 411  $\text{cr}_D(a_1a_3) = 2$ . For  $i, j \in \{0, 1, 2, 3\}$ ,  $i \neq j$ , let  $P_{ij}$  be the antiroute guaran-  
 412 teed by Lemma 10. It is easy to verify that the only antiroute of size 1 from  
 413  $\pi_0$  to  $\pi_1$  is  $\{(01)\}$ , and so necessarily  $P_{01} = \{(01)\}$ . Analogous arguments  
 414 show that necessarily  $P_{23} = \{(01)\}$  and that  $P_{12} = P_{03} = \{(23)\}$ . It is also  
 415 readily checked that there are two antiroutes of size 2 from  $\pi_0$  to  $\pi_2$ , namely  
 416  $\{(04), (14)\}$  and  $\{(24), (34)\}$  (moreover, these are also the two antiroutes of  
 417 size 2 from  $\pi_1$  to  $\pi_3$ ). Thus each of  $P_{02}$  and  $P_{13}$  is either  $\{(04), (14)\}$  or  
 418  $\{(24), (34)\}$ .

FIGURE 8. This cannot be the key of a clean drawing of  $K_{5,n}$ .

Thus  $P_{01}, P_{03}, P_{12}$ , and  $P_{23}$  are all determined:

$$P_{01} = P_{23} = \{(01)\}, P_{03} = P_{12} = \{(23)\},$$

and there are four possible combinations of  $P_{02}$  and  $P_{13}$ :

(a)  $P_{02} = P_{13} = \{(04), (14)\}$ .

422

In this case, by Lemma 10,  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{04} = \{(a_0a_2), (a_1a_3)\}$ ,  $Q_{14} = \{(a_0a_2), (a_1a_3)\}$ ,  $Q_{23} = \{(a_0a_3), (a_1a_2)\}$ , and  $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$ .

(b)  $P_{02} = \{(04), (14)\}$  and  $P_{13} = \{(24), (34)\}$ .

427

In this case, by Lemma 10,  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{04} = Q_{14} = \{(a_0a_2)\}$ ,  $Q_{23} = \{(a_0a_3), (a_1a_2)\}$ ,  $Q_{24} = Q_{34} = \{(a_1a_3)\}$ , and  $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$ .

(c)  $P_{02} = \{(24), (34)\}$  and  $P_{13} = \{(04), (14)\}$ .

432

In this case, by Lemma 10,  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{04} = Q_{14} = \{(a_1a_3)\}$ ,  $Q_{23} = \{(a_0a_3), (a_1a_2)\}$ ,  $Q_{24} = Q_{34} = \{(a_0a_2)\}$ , and  $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$ .

(d)  $P_{02} = P_{13} = \{(24), (34)\}$ .

437

In this case, by Lemma 10,  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ ,  $Q_{23} = \{(a_0a_3), (a_1a_2)\}$ ,  $Q_{24} = Q_{34} = \{(a_0a_2), (a_1a_3)\}$ , and  $Q_{02} = Q_{03} = Q_{04} = Q_{12} = Q_{13} = Q_{14} = \emptyset$ .

We only analyze (that is, derive a contradiction from) (a). The cases (b), (c), and (d) are handled analogously.

Since  $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$ , it follows that  $\text{rot}_D(2)$  and  $\text{rot}_D(3)$  are equal to each other, and equal to the reverse of each of  $\text{rot}_D(0)$ ,  $\text{rot}_D(1)$ , and  $\text{rot}_D(4)$ . Thus  $\text{rot}_D(0) = \text{rot}_D(1) = \text{rot}_D(4)$ . Since  $Q_{01} = \{(a_0a_1), (a_2a_3)\}$  and  $Q_{04} = \{(a_0a_2), (a_1a_3)\}$ , it follows that in  $\text{rot}_D(0)$ : (i)  $a_0$  and  $a_1$  must be adjacent; (ii)  $a_2$  and  $a_3$  must be adjacent; (iii)  $a_0$  and  $a_2$  must be adjacent; and (iv)  $a_1$  and  $a_3$  must be adjacent. Thus  $\text{rot}_D(0)$  is either  $(a_0a_2a_3a_1)$  or  $(a_0a_1a_3a_2)$ . Now since  $Q_{23} = \{(a_0a_3), (a_1a_2)\}$ , it follows that in  $\text{rot}_D(2)$  (and hence in its reverse  $\text{rot}_D(0)$ ) we have that  $a_0$  is adjacent to  $a_3$ , and that  $a_1$  is adjacent to  $a_2$ . But this is impossible, since in neither  $(a_0a_2a_3a_1)$  nor  $(a_0a_1a_3a_2)$  any of these adjacencies occurs.  $\square$

## 8. PROPERTIES OF CORES. I. FORBIDDEN SUBGRAPHS.

We recall that the *core* of a clean drawing  $D$  of  $K_{5,n}$  is the subgraph  $\Phi^1(D)$  of  $\Phi(D)$  that consists of all the vertices of  $\Phi(D)$  and the edges of  $\Phi(D)$  with label 1. Note that while  $\Phi(D)$  is obviously connected,  $\Phi^1(D)$  may be disconnected. As all edges of a core are labelled 1, we sometimes omit the reference to the edge labels altogether when working with  $\Phi^1(D)$ .

Our first result on the structure of cores is a workhorse for the next few sections.

**Claim 13.** *If  $\pi_1, \pi_2$  and  $\pi_3$  are distinct rotations for white vertices in a drawing of  $K_{5,n}$ , then there exists at most one rotation  $\pi_0$  such that there is an antiroute of size 1 from  $\pi_0$  to each of  $\pi_1, \pi_2$ , and  $\pi_3$ .*

*Proof.* By way of contradiction, suppose that there exist distinct vertices  $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$  and antiroutes of size 1 from  $\pi_i$  to  $\pi_1, \pi_2$ , and  $\pi_3$ , for  $i = 0$  and 4. For  $j = 1, 2, 3$  the antiroutes from  $\pi_0$  and  $\pi_4$  to  $\pi_j$  induce a route  $P_{04}(j)$  of size two from  $\pi_0$  to  $\pi_4$ . Assume without loss of generality that  $\pi_0 = (01234)$ . Suppose that for some  $j$ , the transpositions in  $P_{04}(j)$  involve (in total) four distinct elements in  $\{0, 1, 2, 3, 4\}$ . It is immediately checked that this implies that  $P_{04}(j)$  is the only route of size 2 from  $\pi_0$  to  $\pi_4$ , and that this in turn implies that at least two of  $\pi_1, \pi_2$ , and  $\pi_3$  are equal to each other, a contradiction. Thus each of  $P_{04}(1), P_{04}(2)$ , and  $P_{04}(3)$  involve fewer than four elements in  $\{0, 1, 2, 3, 4\}$ . None of these routes can involve only two elements (since they have size 2, and  $\pi_0 \neq \pi_4$ ), and so we conclude that each of  $P_{04}(1), P_{04}(2)$ , and  $P_{04}(3)$  involve exactly three elements in  $\{0, 1, 2, 3, 4\}$ . In particular,  $P_{04}(1)$  must equal either  $\{(k, k+1), (k, k+2)\}$  or  $\{(k+1, k+2), (k, k+2)\}$ , for some  $j \in \{0, 1, 2, 3, 4\}$  (operations are modulo 5; we note that we deviate from the usual notation and separate the elements of a transposition with a comma, for readability purposes). We derive a contradiction assuming that the first possibility holds; the other possibility is handled analogously. Relabelling 0, 1, 2, 3, and 4, if needed, we may assume that  $P_{04}(1) = \{(01), (02)\}$ . Thus  $\pi_4$  is (03412). It is readily verified that the only routes of size 2 from  $\pi_0 = (01234)$  to  $\pi_4 = (03412)$  are  $P_{04}(1) = \{(01), (02)\}$  and  $\{(03), (04)\}$ . This in turn immediately implies



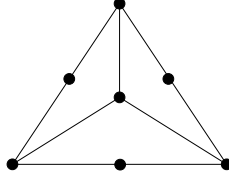


FIGURE 9. The graph obtained by subdividing exactly once each of the edges in a 3-cycle of  $K_4$ .

that the antiroutes of size 1 from  $\pi_0$  to  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$  are either  $\{(01)\}$  or  $\{(04)\}$ , since the transpositions (02) and (03) cannot be applied to  $\pi_0$ . But then we arrive from  $\pi_0$  to two elements in  $\{\pi_1, \pi_2, \pi_3\}$  by applying the same transposition; that is,  $\pi_i = \pi_j$  for some  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ , a contradiction.  $\square$

**Proposition 14.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ . Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free. Then:*

- (1)  $\Phi^1(D)$  does not contain  $K_{2,3}$  as a subgraph.
- (2)  $\Phi^1(D)$  has maximum degree at most 3.
- (3)  $\Phi^1(D)$  does not contain as a subgraph the graph obtained from  $K_4$  by subdividing exactly once each of the edges in a 3-cycle (see Fig. 9).

*Proof.* We start by noting that (1) follows immediately by Claim 13 and Lemma 9.

Suppose now by way of contradiction that  $\Phi^1(D)$  has a vertex  $\pi_0$  of degree at least 4. Thus  $\Phi^1(D)$  has distinct vertices  $\pi_1, \pi_2, \pi_3, \pi_4$  such that the edge joining  $\pi_0$  to  $\pi_i$  has label 1, for  $i = 1, 2, 3, 4$ . Thus, for  $i = 1, 2, 3, 4$ , there exists an antiroute from  $\pi_0$  to  $\pi_i$  of size 1. Without loss of generality we may assume  $\pi_0 = (01234)$ . The five cyclic rotations that have an antiroute of size 1 to  $\pi_0$  are (01432), (03214), (03421), (04312), and (04231). By performing a relabelling  $j \rightarrow j + 1$  on  $\{0, 1, 2, 3, 4\}$  for some  $j \in \{0, 1, 2, 3, 4\}$  (with operations modulo 5) if needed (note that the cyclic permutation  $\pi_0 = (01234)$  is left unchanged in such a relabelling), we may assume without loss of generality that  $\{\pi_1, \pi_2, \pi_3, \pi_4\} = \{(01432), (03214), (03421), (04312)\}$ . By exchanging  $\pi_1, \pi_2, \pi_3, \pi_4$  if needed, we may assume that  $\pi_1 = (01432)$ ,  $\pi_2 = (04312)$ , and  $\pi_3 = (03421)$ .

Since  $\Phi(D)$  is  $\{0, 4\}$ -free, it follows by Proposition 8 that the edge joining  $\pi_i$  to  $\pi_j$  has label 2, for  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ . Thus, for  $i, j = 1, 2, 3$ ,  $i \neq j$ , there exists an antiroute from  $\pi_i$  to  $\pi_j$  of size 2. Thus  $\Phi(D)$  contains as a subgraph the graph in Figure 7, contradicting Proposition 11. This proves (2).

We finally prove (3). Suppose by way of contradiction that  $\Phi^1(D)$  contains as a subgraph the graph obtained from  $K_4$  by subdividing once each of the edges in a 3-cycle (Fig. 9). Let  $\rho_0$  be the “central vertex” in Fig. 9, that is, the only vertex in  $\Phi^1(D)$  adjacent to three degree-3 vertices, and

let  $\rho_1, \rho_3, \rho_4$  denote these three vertices. An argument similar to the one in the second paragraph of this proof shows the following: if  $\rho_0 = (01234)$  is a vertex adjacent to vertices  $\rho_1, \rho_3, \rho_4$  in  $\Phi^1(D)$ , then we may assume (that is, perhaps after a relabelling of  $0, 1, 2, 3, 4$ ), that  $\rho_1 = (01432)$ ,  $\rho_3 = (04231)$ , and  $\rho_4 = (04312)$ . Now let  $\rho_2$  be the vertex adjacent to  $\rho_1$  and  $\rho_3$  in  $\Phi^1(D)$ . Thus it follows that in  $\Phi(D)$ , the edges joining  $\rho_0$  and  $\rho_1$ ,  $\rho_0$  and  $\rho_3$ ,  $\rho_1$  and  $\rho_2$ , and  $\rho_2$  and  $\rho_3$  are labelled 1. By Proposition 8, the edge joining  $\rho_1$  and  $\rho_3$ , as well as the edge joining  $\rho_0$  and  $\rho_2$  have even labels, which must be 2 since  $\Phi(D)$  is  $\{0, 4\}$ -free. Now it is easy to verify that the only cyclic permutation other than  $\rho_0$  which has antiroutes of size 1 to both  $\rho_1$  and  $\rho_3$  is  $(03241)$ . Thus  $\rho_2$  must be  $(03241)$ . But then the subgraph of  $\Phi(D)$  induced by  $\rho_0, \rho_1, \rho_2$ , and  $\rho_3$  is isomorphic to the graph in Figure 8, contradicting Proposition 12.  $\square$

## 9. PROPERTIES OF CORES. II. STRUCTURAL PROPERTIES.

**Proposition 15.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free. Then:*

- (1)  $\Phi^1(D)$  is bipartite.
- (2)  $\Phi^1(D)$  is connected.

*Proof.* Suppose that  $C = (\pi_0, \pi_1, \pi_2, \dots, \pi_{r-1}, \pi_r, \pi_0)$  is an odd cycle in  $\Phi^1(D)$ . It follows from Proposition 8 that  $\pi_0\pi_2$  must have an even label in  $\Phi(D)$ , since  $\pi_0\pi_1$  and  $\pi_1\pi_2$  are both labelled 1 in  $\Phi(D)$ ; now this even label must be 2, since  $\Phi(D)$  is  $\{0, 4\}$ -free. Similarly, since  $\pi_2\pi_3$  and  $\pi_3\pi_4$  are also labelled 1 in  $\Phi(D)$ , then  $\pi_2\pi_4$  must also be labelled 2 in  $\Phi(D)$ . Now since both  $\pi_0\pi_2$  and  $\pi_2\pi_4$  have label 2 in  $\Phi(D)$ , it follows that  $\pi_0\pi_4$  also has label 2 in  $\Phi(D)$ . By repeating this argument we find that  $\pi_0\pi_j$  must have label 2 in  $\Phi(D)$  for every even  $j$ . In particular,  $\pi_0\pi_r$  must have label 2, contradicting that  $\pi_0\pi_r$  is in  $\Phi^1(D)$  (that is, that the label of  $\pi_0\pi_r$  in  $\Phi(D)$  is 1). Thus  $\Phi^1(D)$  cannot have an odd cycle. This proves (1).

To prove (2) we assume, by way of contradiction, that  $\Phi^1(D)$  is not connected.

We start by observing that  $\Phi(D)$  must have at least one edge labelled 1. Indeed, otherwise every edge  $\Phi(D)$  has label of at least 2, and so  $\text{cr}(D) \geq 2\binom{n}{2} = n(n-1) > Z(5, n)$ , contradicting the optimality of  $D$ .

Thus there exists a component  $H$  of  $\Phi^1(D)$  with at least 2 vertices. Let  $U$  be the set of white vertices whose rotation is a vertex in  $H$ , and let  $V$  be all the other white vertices. Let  $r := |U|$  and  $s := |V|$ . Note that

$$\begin{aligned} \text{cr}(D) &= \sum_{\substack{a_i, a_j \in U, \\ a_i \neq a_j}} \text{cr}_D(a_i, a_j) + \sum_{\substack{a_i, a_j \in V, \\ a_i \neq a_j}} \text{cr}_D(a_i, a_j) + \sum_{a_i \in U, a_j \in V} \text{cr}_D(a_i, a_j) \\ (1) \quad &\geq Z(5, r) + Z(5, s) + 2rs, \end{aligned}$$

since every vertex of  $U$  is joined to every vertex of  $V$  by an edge with a label 2 or greater.

We claim that, moreover, strict inequality must hold in (1). To see this, first we note that, since  $H$  has at least 2 vertices, it follows that there exist white vertices  $a_k, a_\ell$  whose rotations are in  $H$  and such that  $\text{cr}_D(a_k, a_\ell) = 1$ . Since by assumption  $\Phi^1(D)$  is not connected, there is a vertex  $\pi$  in  $\Phi^1(D)$  not in  $H$ . Let  $a_i$  be a white vertex such that  $\text{rot}_D(a_i) = \pi$ . Now  $\text{cr}_D(a_k, a_i)$  and  $\text{cr}_D(a_\ell, a_i)$  are both at least 2. However, we cannot have  $\text{cr}_D(a_k, a_i)$  and  $\text{cr}_D(a_\ell, a_i)$  both equal to 2, since then  $\text{cr}_D(a_k, a_\ell) = 1$  would contradict Proposition 7. Thus either  $\text{cr}_D(a_k, a_i)$  or  $\text{cr}_D(a_\ell, a_i)$  is at least 3. This proves that Inequality (1) must be strict, that is,

$$(2) \quad \text{cr}(D) > Z(5, r) + Z(5, s) + 2rs.$$

Suppose that  $r$  (and consequently, also  $s$ ) is even. In this case, since  $Z(5, m) = m(m-2)$  for even  $m$ , using (2) we obtain  $\text{cr}(D) > r(r-2) + s(s-2) + 2rs = (r+s)(r+s-2) = Z(5, r+s) = Z(5, n)$ , contradicting the optimality of  $D$ .

Suppose finally that  $r$  is odd (and so  $s$  is odd, since  $|U| + |V| = n$  is even). Using that  $r$  and  $s$  are odd, and that  $Z(5, m) = (m-1)^2$  for odd  $m$ , with (2) we obtain  $\text{cr}(D) > (r-1)^2 + (s-1)^2 + 2rs = (r+s)(r+s-2) + 2 = Z(5, r+s) + 2 = Z(5, n) + 2$ , again contradicting the optimality of  $D$ . This finishes the proof of (2).  $\square$

### 10. PROPERTIES OF CORES. III. MINIMUM DEGREE.

**Proposition 16.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free. Let  $\pi_0, \pi_1, \pi_2, \pi_3$  be a path in  $\Phi^1(D)$ . Suppose that in  $\Phi^1(D)$ ,  $\pi_1$  is the only vertex adjacent to both  $\pi_0$  and  $\pi_2$ , and  $\pi_2$  is the only vertex adjacent to both  $\pi_1$  and  $\pi_3$ . Then:*

- (1) *every vertex in  $\Phi^1(D)$  is adjacent (in  $\Phi^1(D)$ ) to a vertex in  $\{\pi_0, \pi_1, \pi_2, \pi_3\}$ ; and*
- (2)  *$\pi_0$  and  $\pi_3$  are adjacent in  $\Phi^1(D)$ .*

*Proof.* Let  $\pi_0, \pi_1, \dots, \pi_{r-1}$  be the vertices of  $\Phi^1(D)$  (and of  $\Phi(D)$  as well). For  $i, j \in [r], i \neq j$ , let  $\lambda_{ij}$  denote the label of the edge that joins  $\pi_i$  to  $\pi_j$  in  $\Phi(D)$ . Recall that  $\Phi^1(D)$  is bipartite (Proposition 15(1)). Since  $\pi_0, \pi_1, \pi_2, \pi_3$  is a path in  $\Phi(D)$ , it follows that  $\pi_0$  and  $\pi_2$  are in the same chromatic class  $A$ , and  $\pi_1$  and  $\pi_3$  are in the same chromatic class  $B$ . Moreover, since  $\Phi(D)$  is  $\{0, 4\}$ -free, it follows from Proposition 8 that  $\lambda_{ij} = 2$  whenever  $\pi_i$  and  $\pi_j$  belong to the same chromatic class. Thus we have  $\lambda_{02} = \lambda_{13} = 2$  and (since  $\pi_0, \pi_1, \pi_2, \pi_3$  is a path in  $\Phi^1(D)$ )  $\lambda_{01} = \lambda_{12} = \lambda_{23} = 1$ . It follows that the equations of  $\mathcal{L}(\Phi(D))$  corresponding to  $\pi_0, \pi_1, \pi_2$ , and  $\pi_3$  are:

$$\begin{aligned} E_0 : \quad & 2t_0 - t_1 + (\lambda_{03} - 2)t_3 + \sum_{j \in [r], j > 3} (\lambda_{0j} - 2)t_j = 0, \\ E_1 : \quad & -t_0 + 2t_1 - t_2 + \sum_{j \in [r], j > 3} (\lambda_{1j} - 2)t_j = 0, \\ E_2 : \quad & -t_1 + 2t_2 - t_3 + \sum_{j \in [r], j > 3} (\lambda_{2j} - 2)t_j = 0, \\ E_3 : \quad & (\lambda_{03} - 2)t_0 - t_2 + 2t_3 + \sum_{j \in [r], j > 3} (\lambda_{3j} - 2)t_j = 0, \end{aligned}$$

where for simplicity we define  $E_i := E(\pi_i, \Phi(D))$  for  $i \in \{0, 1, 2, 3\}$ . Summing up these four linear equations we obtain

$$(3) \quad (\lambda_{03} - 1)t_0 + (\lambda_{03} - 1)t_3 + \sum_{j \in [r], j > 3} (\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} - 8)t_j = 0$$

We claim all the coefficients in (3) are nonnegative. First we note that since  $\lambda_{03} \geq 1$ , then the coefficients of  $t_0$  and  $t_3$  are indeed nonnegative. For the remaining coefficients, consider any vertex  $\pi_j$  in  $\Phi(D)$ , with  $j > 3$ . Since  $\Phi(D)$  is  $\{0, 4\}$ -free, it follows that  $\lambda_{ij} \geq 1$  for every  $i \in \{0, 1, 2, 3\}$ .

Since  $\Phi^1(D)$  is bipartite, it follows that  $\pi_j$  cannot be adjacent (in  $\Phi^1(D)$ ) to two elements in  $\{\pi_0, \pi_1, \pi_2, \pi_3\}$  whose indices have distinct parity. Now it follows by hypothesis that  $\pi_j$  cannot be adjacent to both  $\pi_0$  and  $\pi_2$ , or to  $\pi_1$  and  $\pi_3$ . Thus  $\pi_j$  is adjacent to at most one of  $\pi_0, \pi_1, \pi_2$  and  $\pi_3$  in  $\Phi^1(D)$ . Using this, and the fact that  $\pi_j$  has the same chromatic class as exactly two of these vertices, it follows that at least one element in  $\{\lambda_{0j}, \lambda_{1j}, \lambda_{2j}, \lambda_{3j}\}$  is 3, and at least two elements are 2. Thus it follows that  $(\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} - 8) \geq 0$ .

Therefore (3) implies that  $(\lambda_{03} - 1)t_0 + (\lambda_{03} - 1)t_3 \leq 0$ . Recall that  $\lambda_{03}$  is either 1 or 3. If  $\lambda_{03} = 3$ , then we have  $2t_0 + 2t_3 \leq 0$ , which contradicts (Proposition 6) that  $\mathcal{L}(\Phi(D))$  has a positive integral solution. We conclude that  $\lambda_{03} = 1$ , that is,  $\pi_0$  and  $\pi_3$  are adjacent in  $\Phi^1(D)$ . This proves (2).

We also note that since  $\lambda_{03} = 1$ , (3) implies that

$$(4) \quad \sum_{j \in [r], j > 3} (\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} - 8)t_j = 0.$$

By way of contradiction suppose there is a vertex  $\pi_4$  adjacent to none of  $\pi_0, \pi_1, \pi_2, \pi_3$  in  $\Phi^1(D)$ . Then each of  $\lambda_{04}, \lambda_{14}, \lambda_{24}, \lambda_{34}$  is at least 2. Using Proposition 8 and that  $\Phi(D)$  is  $\{0, 4\}$ -free, it follows that two of these  $\lambda$ s are 2, and the other two are 3. Therefore  $(\lambda_{04} + \lambda_{14} + \lambda_{24} + \lambda_{34} - 8) = 2$ . Using (4) we obtain

$$(5) \quad 2t_4 + \sum_{j \in [r], j > 4} (\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} - 8)t_j = 0.$$

We recall that  $\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{3j} - 8 \geq 0$  for every  $j > 3$ . Using this and (5), it follows that  $2t_4 \leq 0$ . But this contradicts that  $\mathcal{L}(\Phi(D))$  has a positive integral solution.  $\square$

**Proposition 17.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free. Then  $\Phi^1(D)$  has minimum degree at least 2.*

*Proof.* By way of contradiction, suppose that  $\Phi^1(D)$  has a vertex of degree 0 or 1.

Suppose first that  $\Phi^1(D)$  has a vertex of degree 0. Then the connectedness of  $\Phi^1(D)$  implies that this is the only vertex in  $\Phi^1(D)$  (and, consequently, the only vertex in  $\Phi(D)$ ). Thus all vertices of  $D$  have the same rotation. Since if  $a_i, a_j$  have the same rotation in a drawing  $D'$  then  $\text{cr}_{D'}(a_i, a_j) = 4$ ,

617 it follows that  $\text{cr}(D) \geq 4\binom{n}{2} = 2n(n-1)$ . Since  $Z(5, n) = n(n-2)$  and  $D$   
 618 is optimal, we must have  $2n(n-1) \leq n(n-2)$ , but this inequality does not  
 619 hold for any positive integer  $n$ .

620 Thus we may assume that  $\Phi^1(D)$  has a vertex of degree 1.

621 Let  $\pi_0, \pi_1, \dots, \pi_{m-1}$  denote the vertices of  $\Phi^1(D)$ . Without any loss of  
 622 generality we may assume that  $\pi_0$  has degree 1 in  $\Phi^1(D)$ . For  $i, j \in [m]$ , let  
 623  $\lambda_{ij}$  denote the label of the edge  $\pi_i\pi_j$ .

624 We divide the rest of the proof into two cases.

625 CASE 1.  $\Phi^1(D)$  has a path with 4 vertices starting at  $\pi_0$ .

626 Without loss of generality, let  $\pi_0, \pi_1, \pi_2, \pi_3$  be this path. Since  $\pi_0$  is a  
 627 leaf, it follows that  $\pi_1$  is the only vertex of  $\Phi^1(D)$  adjacent to both  $\pi_0$  and  
 628  $\pi_2$ . We note that then there must be a vertex in  $\Phi^1(D)$  (say  $\pi_4$ , without  
 629 loss of generality) adjacent to both  $\pi_1$  and  $\pi_3$ , as otherwise it would follow  
 630 by Proposition 16(2) that  $\pi_0$  is adjacent to  $\pi_3$ , contradicting that  $\pi_0$  is a  
 631 leaf. Thus  $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$  is a cycle.

632 For  $i, j \in [5]$ , let  $\lambda_{ij}$  denote the label of  $\pi_i\pi_j$  in  $\Phi(D)$ . Since the edges  
 633  $\pi_0\pi_1, \pi_1\pi_2, \pi_2\pi_3, \pi_3\pi_4$  and  $\pi_1\pi_4$  are all in  $\Phi^1(D)$ , it follows that  $\lambda_{01} = \lambda_{12} =$   
 634  $\lambda_{23} = \lambda_{34} = \lambda_{14} = 1$ . Now since  $\Phi(D)$  is  $\{0, 4\}$ -free, using Proposition 8 it  
 635 follows that  $\lambda_{02} = \lambda_{04} = \lambda_{24} = \lambda_{13} = 2$  and (since  $\pi_0\pi_3$  is not in  $\Phi^1(D)$ )  
 636 that  $\lambda_{03} = 3$ .

637 SUBCASE 1.1.  $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$  are all the vertices in  $\Phi^1(D)$ .

638 In this case the linear system  $\mathcal{L}(\Phi(D))$  reads:

$$\begin{array}{rclclclcl} E_0 & : & 2t_0 & - & t_1 & & + & t_3 & & = & 0, \\ E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & - & t_4 & = & 0, \\ E_2 & : & & - & t_1 & + & 2t_2 & - & t_3 & & = & 0, \\ E_3 & : & t_0 & & & - & t_2 & + & 2t_3 & - & t_4 & = & 0, \\ E_4 & : & & - & t_1 & & & - & t_3 & + & 2t_4 & = & 0, \end{array}$$

639 where for brevity we let  $E_i := E(\pi_i, \Phi(D))$  for  $i \in [5]$ .

640 Subtracting  $E_4$  from  $E_2$ , we obtain that  $t_2 = t_4$ . Adding the equations  
 641  $E_0, E_1, E_2$ , and using  $t_2 = t_4$ , we obtain  $t_0 = 0$ . Thus the system  $\mathcal{L}(\Phi(D))$   
 642 has no positive integral solution, contradicting (by Proposition 6) the opti-  
 643 mality of  $D$ .

644 SUBCASE 1.2.  $\Phi^1(D)$  has a vertex  $\pi_5 \notin \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ .

645 The connectedness of  $\Phi^1(D)$  implies that  $\pi_5$  is adjacent to  $\pi_i$  for some  
 646  $i \in \{0, 1, 2, 3, 4\}$ . Since  $\pi_0$  is a leaf only adjacent to  $\pi_1$ , then  $i \neq 0$ . Since  $\pi_1$   
 647 already has degree 3 in  $\Phi^1(D)$ , it follows from Proposition 14(2) that  $i \neq 1$ .  
 648 Thus  $i$  is either 2, 3 or 4. Since the roles of 2 and 4 are symmetric, we may  
 649 conclude that  $\pi_5$  is adjacent to either  $\pi_2$  or to  $\pi_3$ .

650 Suppose first that  $\pi_5$  is adjacent to  $\pi_3$  in  $\Phi^1(D)$ .

In this case  $\lambda_{35} = 1$ . Using Proposition 8, that  $\Phi(D)$  is  $\{0, 4\}$ -free, that  $\pi_0$  is only adjacent to  $\pi_1$ , and Claim 13, we obtain  $\lambda_{05} = \lambda_{25} = \lambda_{45} = 2$  and that  $\lambda_{15} = 3$ . Thus in this case the 0-th and the 5-th equations of the system  $\mathcal{L}(\Phi(D))$  read:

$$\begin{aligned} E_0 &: 2t_0 - t_1 + t_3 + \sum_{j \in [m], j > 5} (\lambda_{0j} - 2)t_j = 0. \\ E_5 &: \quad + t_1 - t_3 + 2t_5 + \sum_{j \in [m], j > 5} (\lambda_{5j} - 2)t_j = 0. \end{aligned}$$

651 where for brevity we let  $E_i := E(\pi_i, \Phi(D))$  for  $i = 0$  and 5.

652 Adding these equations, we get

$$(6) \quad 2t_0 + 2t_5 + \sum_{j \in [m], j > 5} (\lambda_{0j} + \lambda_{5j} - 4)t_j = 0.$$

We now argue that  $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$  whenever  $j > 5$ . To see this, note that  $\pi_0$  and  $\pi_5$  are in the same chromatic class. If  $\pi_j$  is in the same chromatic class, then, since  $\Phi(D)$  is  $\{0, 4\}$ -free, it follows that  $\lambda_{0j}$  and  $\lambda_{5j}$  are both 2, and so  $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$ , as claimed. If  $\pi_j$  is in the other chromatic class, then both  $\lambda_{0j}$  and  $\lambda_{5j}$  are odd. Since  $\pi_0$  is a leaf whose only adjacent vertex is  $\pi_1$ , it follows that  $\lambda_{0j} = 3$ . On the other hand,  $\lambda_{5j}$  is either 1 or 3. In particular,  $\lambda_{5j} \geq 1$ , and thus also in this case  $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$ , as claimed. It follows from this observation and (6) that

$$2t_0 + 2t_5 \leq 0,$$

653 and so the system  $\mathcal{L}(\Phi(D))$  has no positive integral solution, contradicting  
654 Proposition 6.

655 Suppose finally that  $\pi_5$  is adjacent to  $\pi_2$  in  $\Phi^1(D)$ .

656 Consider then the path  $\pi_0, \pi_1, \pi_2, \pi_5$ . Since  $\pi_0$  is a leaf, it follows that  $\pi_1$   
657 is the only vertex adjacent to both  $\pi_0$  and  $\pi_2$ . Now note that  $\pi_2$  is the only  
658 vertex adjacent to both  $\pi_1$  and  $\pi_5$ , since by Proposition 14(2)  $\pi_1$  cannot  
659 be incident to any vertex other than  $\pi_0, \pi_2$ , and  $\pi_4$ . Thus Proposition 16  
660 applies, and so we must have that  $\pi_0$  and  $\pi_5$  are adjacent in  $\Phi^1(D)$ . But  
661 this is impossible, since the only vertex in  $\Phi^1(D)$  adjacent to the leaf  $\pi_0$  is  
662  $\pi_1$ .

663 *CASE 2.  $\Phi^1(D)$  has no path with 4 vertices starting at  $\pi_0$ .*

664 We recall that  $\pi_0$  is a leaf in  $\Phi^1(D)$ . Let  $\pi_1$  be the vertex adjacent to  $\pi_0$ .

665 Suppose first that  $\pi_0$  and  $\pi_1$  are the only vertices in  $\Phi^1(D)$ . Then  
666  $\mathcal{L}(\Phi(D))$  consists of only two equations, namely  $2t_1 - t_0 = 0$  and  $2t_0 - t_1 =$   
667  $0$ . This system obviously has no positive integral solutions, contradicting  
668 Proposition 6.

669 We may then assume that there is an additional vertex  $\pi_2$  in  $\Phi^1(D)$ . By  
670 connectedness of  $\Phi^1(D)$ , and since  $\pi_0$  is a leaf, it follows that  $\pi_2$  is adjacent  
671 to  $\pi_1$ .

672 If  $\pi_0, \pi_1, \pi_2$  are the only vertices  $\Phi(D)$ , then the system  $\mathcal{L}(\Phi(D))$  consists  
673 of the three equations  $2t_0 - t_1 = 0$ ,  $-t_0 + 2t_1 - t_2 = 0$ , and  $-t_1 + 2t_2 = 0$ .

674 Adding these equations we obtain  $t_0 + t_2 = 0$ . Thus also in this case  $\mathcal{L}(\Phi(D))$   
 675 does not have a positive integral solution, again contradicting Proposition 6.  
 676 Thus there must exist an additional vertex  $\pi_3$  in  $\Phi^1(D)$ . Since  $\pi_0$  is a  
 677 leaf, and by assumption (we are working in Case 2) there is no path with 4  
 678 vertices starting at  $\pi_0$ , it follows that  $\pi_3$  must be adjacent to  $\pi_1$ . We already  
 679 know that  $\lambda_{01} = \lambda_{12} = \lambda_{13} = 1$ . Since  $\Phi(D)$  is  $\{0, 4\}$  free, it follows from  
 680 Proposition 8 that  $\lambda_{02} = \lambda_{03} = \lambda_{23} = 2$ . Thus in this case  $\mathcal{L}(\Phi(D))$  consists  
 681 of the equations  $2t_0 - t_1 = 0$ ,  $-t_0 + 2t_1 - t_2 - t_3 = 0$ ,  $-t_1 + 2t_2 = 0$ , and  
 682  $-t_1 + 2t_3 = 0$ . It is an elementary exercise to show that these equations  
 683 do not have a simultaneous positive integral solution, and so in this case we  
 684 also obtain a contradiction to Proposition 6.  $\square$

#### 685 11. PROPERTIES OF CORES. IV. GIRTH AND MAXIMUM SIZE.

686 **Proposition 18.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free. Then:*

- 688 (1)  $\Phi^1(D)$  has girth 4.
- 689 (2) If  $v$  is a degree-2 vertex in  $\Phi^1(D)$ , then  $v$  is in a 4-cycle in  $\Phi^1(D)$ .
- 690 (3)  $\Phi^1(D)$  has at most 7 vertices.

691 *Proof.* By Proposition 17, the minimum degree of  $\Phi^1(D)$  is at least 2. Since  
 692  $\Phi^1(D)$  is simple and bipartite, it immediately follows that the girth of  $\Phi^1(D)$   
 693 is a positive number greater than or equal to 4. Let  $\pi_0, \pi_1, \pi_2, \pi_3$  be a path  
 694 in  $\Phi^1(D)$ . If there is a vertex other than  $\pi_1$  adjacent to both  $\pi_0$  and  $\pi_2$ , or  
 695 a vertex other than  $\pi_2$  adjacent to both  $\pi_1$  and  $\pi_3$ , then  $\Phi^1(D)$  clearly has a  
 696 4-cycle, and we are done. Otherwise, it follows from Proposition 16(2) that  
 697  $\pi_0$  is adjacent to  $\pi_3$ , and so  $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$  is a 4-cycle. Thus (1) follows.

698 Now let  $\pi_1$  be a degree-2 vertex in  $\Phi^1(D)$ . Since  $\Phi^1(D)$  has minimum  
 699 degree at least 2, using (1) it obviously follows that there exists a path  
 700  $\pi_0, \pi_1, \pi_2, \pi_3$  in  $\Phi^1(D)$ . If there is a vertex adjacent to both  $\pi_0$  and  $\pi_2$  other  
 701 than  $\pi_1$ , then  $\pi_1$  is obviously contained in a 4-cycle. In such a case we are  
 702 done, so suppose that this is not the case. Since  $\pi_1$  is only adjacent to  $\pi_0$   
 703 and  $\pi_2$ , using that the degree of  $\pi_1$  is 2 it follows that no vertex other than  
 704  $\pi_2$  is adjacent to both  $\pi_1$  and  $\pi_3$ . Thus it follows from Proposition 16(2)  
 705 that  $\pi_0$  and  $\pi_3$  are adjacent in  $\Phi^1(D)$ . Thus  $\pi_1$  is contained in the 4-cycle  
 706  $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ , and (2) follows.

707 Let  $C = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$  be a 4-cycle in  $\Phi^1(D)$ ; the existence of  $C$  is  
 708 guaranteed from (1). By Proposition 14(1)  $\Phi^1(D)$  contains no subgraph  
 709 isomorphic to  $K_{2,3}$ , and so, in  $\Phi^1(D)$ , no vertex other than  $\pi_1$  or  $\pi_3$  is  
 710 adjacent to both  $\pi_0$  and  $\pi_2$ , and no vertex other than  $\pi_2$  or  $\pi_0$  is adjacent to  
 711 both  $\pi_1$  and  $\pi_3$ . Thus Proposition 16 applies. Using Proposition 14(2) and  
 712 Proposition 16(1), we obtain that  $\Phi^1(D)$  has at most 4 vertices other than  
 713  $\pi_0, \pi_1, \pi_2$ , and  $\pi_3$ ; that is,  $\Phi^1(D)$  has at most 8 vertices in total; moreover,  
 714 if  $\Phi^1(D)$  has exactly 8 vertices, then every vertex of  $C$  has degree 3. Since  
 715  $C$  was an arbitrary 4-cycle, we have actually proved that if  $\Phi^1(D)$  has 8

vertices, then every vertex contained in a 4-cycle must have degree 3. In view of (2), this implies that if  $\Phi^1(D)$  has 8 vertices, then it must be cubic.

Now the unique (up to isomorphism) cubic connected bipartite graph on 8 vertices is the 3-cube. Since the 3-cube contains as a subgraph the graph in Figure 9, it follows that  $\Phi^1(D)$  cannot have exactly 8 vertices.  $\square$

## 12. THE POSSIBLE CORES OF AN ANTIPODAL-FREE OPTIMAL DRAWING.

Our goal in this section is to establish Lemma 21, which states that the core of every antipodal-free optimal drawing of  $K_{5,n}$  is isomorphic to either a 4-cycle or to the graph  $\overline{C}_6$  obtained from the 6-cycle by adding an edge joining two diametrically oposed vertices (see Figure 10).

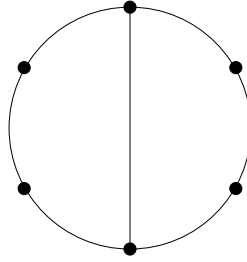


FIGURE 10. The graph  $\overline{C}_6$ .

We first show this for the particular case in which  $\Phi(D)$  is not only antipodal-free (that is, 0-free), but also 4-free:

**Proposition 19.** *Let  $D$  be an optimal drawing of  $K_{5,n}$ , with  $n$  even. If  $\Phi(D)$  is  $\{0, 4\}$ -free, then  $\Phi^1(D)$  is isomorphic to the 4-cycle or to  $\overline{C}_6$ .*

*Proof.* By way of contradiction, suppose that  $\Phi^1(D)$  is isomorphic to neither a 4-cycle nor to  $\overline{C}_6$ . Recall that  $\Phi^1(D)$  has minimum degree at least 2 (Proposition 17). We divide the proof into two cases, depending on whether or not  $\Phi^1(D)$  has degree-2 vertices.

CASE 1.  $\Phi^1(D)$  has at least one degree-2 vertex.

By Proposition 18(3),  $\Phi^1(D)$  has at most 7 vertices. If all the vertices in  $\Phi^1(D)$  have degree 2, then (since  $\Phi^1(D)$  is simple and, by Proposition 15(2), connected)  $\Phi^1(D)$  is a cycle. By Proposition 18(1), in this case  $\Phi^1(D)$  is a 4-cycle, contradicting our assumption at the beginning of the proof.

Thus we may assume that  $\Phi^1(D)$  has at least one degree-3 vertex. Let  $H$  be the graph obtained by suppressing the degree-2 vertices from  $\Phi^1(D)$ . We call the vertices of  $\Phi^1(D)$  that correspond to the vertices in  $H$  (that is, the degree-3 vertices of  $\Phi^1(D)$ ) the *nodes* of  $\Phi^1(D)$ .



743 It follows from elementary graph theory that  $\Phi^1(D)$  has an even number  
 744 of nodes. Since  $\Phi^1(D)$  has at most 7 vertices, it follows that  $\Phi^1(D)$  has  
 745 either 2, 4, or 6 nodes.

746 SUBCASE 1.1.  $\Phi^1(D)$  has 6 nodes.

747 Up to isomorphism, there are only two cubic simple graphs on 6 nodes,  
 748 namely  $K_{3,3}$  and the triangular prism  $T_3$  (this is the simple cubic graph  
 749 with a matching whose removal leaves two disjoint 3-cycles). Now  $T_3$  has  
 750 two vertex disjoint 3-cycles, and so in order to turn it into a bipartite graph,  
 751 we must subdivide at least 2 edges, that is, add at least two vertices to  $T_3$ .  
 752 Since  $\Phi^1(D)$  has at most 7 vertices, it follows that  $H$  cannot be isomorphic  
 753 to  $T_3$ .

754 Suppose finally that  $H$  is isomorphic to  $K_{3,3}$ . Since no bipartite graph  
 755 on 7 vertices is a subdivision of  $K_{3,3}$ , it follows that  $\Phi^1(D)$  must be itself  
 756 isomorphic to  $K_{3,3}$ . Since  $K_{3,3}$  obviously contains  $K_{2,3}$  as a subgraph, this  
 757 contradicts Proposition 14(1).

758 SUBCASE 1.2.  $\Phi^1(D)$  has 4 nodes.

759 In this case  $H$  must be isomorphic to  $K_4$ , the only cubic graph on four  
 760 vertices. It is readily seen that there are only two ways to turn  $K_4$  into  
 761 a bipartite graph using at most three edge subdivisions. One way is to  
 762 subdivide once each of the edges in a 3-cycle of  $K_4$ , and the other way is  
 763 to subdivide (once) two nonadjacent edges (in the latter case, we obtain a  
 764 graph that has a subgraph isomorphic to  $K_{2,3}$ ). By Proposition 14, neither  
 765 of these graphs can be the core of  $D$ .

766 SUBCASE 1.3.  $\Phi^1(D)$  has 2 nodes.

767 In this case  $H$  must consist of two vertices joined by three parallel edges.  
 768 Since  $\Phi^1(D)$  is bipartite it follows that each of these edges must be sub-  
 769 divided the same number of times modulo 2 (subdividing an edge 0 times  
 770 being a possibility). Moreover, since  $\Phi^1(D)$  is simple at least two edges  
 771 must be subdivided at least once each.

772 Now no edge may be subdivided more than twice, as in this case the  
 773 result would be a graph with a degree-2 vertex belonging to no 4-cycle,  
 774 contradicting Proposition 18(2).

775 Suppose now that some edge of  $H$  is subdivided exactly twice. Then, since  
 776  $\Phi^1(D)$  has at most 7 vertices, it follows that two edges of  $H$  are subdivided  
 777 exactly twice, and the other edge of  $H$  is not subdivided. Thus it follows that  
 778 in this case  $\Phi^1(D)$  is isomorphic to  $\overline{C}_6$ , contradicting our initial assumption.

779 Suppose finally that no edge of  $H$  is subdivided more than once. Since  
 780  $\Phi^1(D)$  is bipartite, it follows that every edge of  $H$  must be subdivided

781 exactly once. Thus  $\Phi^1(D)$  is isomorphic to  $K_{2,3}$ , contradicting Proposi-  
 782 tion 14(1).

783 CASE 2.  $\Phi^1(D)$  has no degree-2 vertices.

784 In this case,  $\Phi^1(D)$  is cubic. By Proposition 15,  $\Phi^1(D)$  is bipartite and  
 785 connected. By Proposition 18(3),  $\Phi^1(D)$  has at most 7 vertices. By ele-  
 786 mentary graph theory, since  $\Phi^1(D)$  is cubic, then it has an even number  
 787 of vertices. Since  $\Phi^1(D)$  is simple, it follows that  $\Phi^1(D)$  has either 4 or 6  
 788 vertices.

789 Now there are no simple cubic bipartite graphs on 4 vertices, so  $\Phi^1(D)$   
 790 must have 6 vertices. Up to isomorphism, the only cubic bipartite graph  
 791 on 6 vertices is  $K_{3,3}$ . But  $\Phi^1(D)$  cannot be isomorphic to  $K_{3,3}$ , since by  
 792 Proposition 14(1)  $\Phi^1(D)$  does not contain a subgraph isomorphic to  $K_{2,3}$ .  
 793  $\square$

794 **Proposition 20.** *Let  $D$  be an antipodal-free, optimal drawing of  $K_{5,n}$ , with*  
 795  *$n$  even. Then  $\Phi(D)$  is 4-free.*

796 *Proof.* By way of contradiction, suppose that  $\Phi(D)$  is not 4-free. Then there  
 797 exist distinct rotations  $\pi, \pi'$ , and white vertices  $a_i, a_j$  such that  $\text{rot}_D(a_i) = \pi$   
 798 and  $\text{rot}_D(a_j) = \pi'$ , and  $\text{cr}_D(a_i, a_j) = 4$ .

799 Without loss of generality, suppose that  $\text{cr}_D(a_i) \leq \text{cr}_D(a_j)$ . We move,  
 800 one by one, every vertex  $a_j$  with rotation  $\pi'$  very close to  $a_i$ , so that in  
 801 the resulting drawing  $D'$  we have  $\text{cr}_{D'}(a_j, a_k) = \text{cr}_{D'}(a_i, a_k)$  for every vertex  
 802  $k \notin \{i, j\}$ . It is readily checked that the resulting drawing  $D'$  is also optimal,  
 803 and  $\Phi(D')$  has one fewer edge with label 4 than  $\Phi(D)$ . By repeating this  
 804 process as many times as needed, we arrive to a drawing  $D^o$  such that  $\Phi(D^o)$   
 805 has exactly one edge with label 4 (if  $\Phi(D)$  has exactly one edge with label 4  
 806 to begin with, then we let  $D^o = D$ ). Denote by  $\pi_0, \pi_1$  the vertices of  $\Phi(D^o)$   
 807 whose joining edge has label 4.

808 If we apply the described process one more time to  $D^o$  with  $\pi = \pi_0$  and  
 809  $\pi' = \pi_1$ , we obtain a  $\{0, 4\}$ -free optimal drawing  $E$  of  $K_{5,n}$ . By Proposi-  
 810 tion 19,  $\Phi^1(E)$  contains a 4-cycle  $(\pi_0, \pi_2, \pi_3, \pi_4, \pi_0)$ . Now if we apply the  
 811 process to  $D^o$  with  $\pi = \pi_1$  and  $\pi' = \pi_0$ , then we obtain another  $\{0, 4\}$ -free  
 812 optimal drawing  $F$  of  $K_{5,n}$ . Note that  $\pi_2, \pi_3, \pi_4$  are not affected in the pro-  
 813 cess, and so  $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$  is a 4-cycle in  $\Phi^1(F)$ . Thus it follows that  
 814  $\Phi^1(D^o)$  has two degree-3 vertices  $\pi_2$  and  $\pi_4$ , plus the vertices  $\pi_0, \pi_1, \pi_3$ ,  
 815 each of which is joined to both  $\pi_2$  and  $\pi_4$  with an edge labelled 1. This  
 816 contradicts Claim 13.  
 817  $\square$

818 **Lemma 21.** *Let  $D$  be an antipodal-free, optimal drawing of  $K_{5,n}$ , with  $n$*   
 819 *even. Then  $\Phi^1(D)$  is isomorphic either to the 4-cycle or to  $\bar{C}_6$ .*

820 *Proof.* By Proposition 20,  $\Phi(D)$  is 4-free. By hypothesis  $\Phi(D)$  is also 0-free  
 821 (since  $D$  is antipodal-free), and so  $\Phi(D)$  is  $\{0, 4\}$ -free. The lemma then  
 822 follows by Proposition 19.  $\square$

## 13. PROOF OF THEOREM 1.

We need one final result before moving on to the proof of Theorem 1. In the following statement and its proof, we sometimes use the notation  $(i, j, k, \ell, m)$  for cyclic permutations (that is, we separate the elements with commas, as opposed to our usual practice in which for such a cyclic permutation we would have written  $(ijklm)$ ).

**Proposition 22.** *Let  $D$  be a drawing of  $K_{5,n}$ . Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free, and that  $\Phi^1(D)$  is a 4-cycle  $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ . Suppose that  $\pi_0 = (01234)$ . Then there exists an  $m \in \{0, 1, 2, 3, 4\}$  and a relabelling of  $\{0, 1, 2, 3, 4\}$  that leaves  $\pi_0$  invariant, such that (operations are modulo 5):*

- $\pi_2 = (m, m+1, m+3, m+4, m+2)$ ; and
- $\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$ .

*Proof.* The reverse permutation  $\overline{\pi_0}$  of  $\pi_0$  is  $(43210)$ . Since  $\pi_0\pi_1$  and  $\pi_0\pi_3$  have label 1 in  $\Phi(D)$ , it follows that each of  $\pi_1$  and  $\pi_3$  is obtained from  $\overline{\pi_0}$  by performing one transposition. Thus there exist distinct  $k, m \in \{0, 1, 2, 3, 4\}$  such that  $\{\pi_1, \pi_3\} = \{(k, k+4, k+2, k+3, k+1), (m, m+4, m+2, m+3, m+1)\}$ .

Suppose that  $k = m+3$ . Using a relabelling on  $\{0, 1, 2, 3, 4\}$  that leaves  $(01234)$  invariant, we may assume that  $m = 2$  and  $k = 0$ . Then  $\{\pi_1, \pi_3\} = \{(04231), (03214)\}$ . Now since the edge joining  $\pi_2$  to each of  $\pi_1$  and  $\pi_3$  in  $\Phi(D)$  has label 1, it follows that there are antiroutes of size 1 from  $\pi_2$  to each of  $\pi_1$  and  $\pi_3$ . It is easy to check that the only such possibility is that  $\pi_2 = (04132)$ . Using the relabelling  $j \mapsto j-2$  on  $\{0, 1, 2, 3, 4\}$ , we get  $\{\pi_0, \pi_1, \pi_2, \pi_3\} = \{(01234), (01432), (03241), (04231)\}$ . But then  $\Phi(D)$  is the labelled graph in Fig. 8, contradicting Proposition 12. An analogous contradiction is obtained under the assumption  $k = m+2$ . Thus  $k = m+1$  or  $k = m+4$ .

Suppose that  $k = m+1$ . Thus  $\{\pi_1, \pi_3\} = \{(m+1, m, m+3, m+4, m+2), (m, m+4, m+2, m+3, m+1)\}$ . Using the relabelling  $j \mapsto j-1$  on  $\{0, 1, 2, 3, 4\}$  (which obviously leaves  $(01234)$  invariant), we obtain  $\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m+4, m+3, m+1, m+2, m)\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$ , as required. Finally, since the edge joining  $\pi_2$  to each of  $\pi_1$  and  $\pi_3$  in  $\Phi(D)$  has label 1, it follows that  $\pi_2 = (m, m+1, m+3, m+4, m+2)$ . The case  $k = m+4$  is handled in a totally analogous manner.  $\square$

**Proposition 23.** *Suppose that  $D$  is a drawing of  $K_{5,n}$ . Suppose that  $\Phi(D)$  is  $\{0, 4\}$ -free, and that  $\Phi^1(D)$  is isomorphic to  $\overline{C_6}$ . Let the vertices of  $\Phi^1(D)$  be labeled  $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ , so that  $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$  and  $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$  are 4-cycles. Suppose that  $\pi_0 = (01234)$ . Then there exists an  $m \in \{0, 1, 2, 3, 4\}$  and a relabelling of  $\{0, 1, 2, 3, 4\}$  that leaves  $\pi_0$  invariant, such that (operations are modulo 5):*

- $\pi_3 = (m, m+4, m+3, m+1, m+2)$ ;

866 •  $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((m, m+4, m+2, m+3, m+1), (m, m+1, m+$   
 867  $3, m+4, m+2)), ((m, m+1, m+4, m+3, m+2), (m, m+2, m+$   
 868  $3, m+1, m+4))\}.$

869 *Proof.* By Proposition 22, there exists an  $m \in \{0, 1, 2, 3, 4\}$  such that  $\pi_2 =$   
 870  $(m, m+1, m+3, m+4, m+2)$  and  $\{\pi_1, \pi_3\} = A := \{(m, m+4, m+2, m+$   
 871  $3, m+1), (m, m+4, m+3, m+1, m+2)\}$ . By the same proposition, there  
 872 exists a  $k \in \{0, 1, 2, 3, 4\}$  such that  $\pi_5 = (k, k+1, k+3, k+4, k+2)$  and  
 873  $\{\pi_3, \pi_4\} = B := \{(k, k+4, k+2, k+3, k+1), (k, k+4, k+3, k+1, k+2)\}.$

874 Since  $\pi_2 \neq \pi_5$ , it follows that  $m \neq k$ . Thus  $k$  is either  $m+1, m+2, m+3,$   
 875 or  $m+4$ . Note that if  $k = m+2$  or  $k = m+3$  then  $A \cap B = \emptyset$ , which  
 876 contradicts that  $\{\pi_3\} = A \cap B$ . Thus  $k$  is either  $m+1$  or  $m+4$ .

877 We work out the details for the case  $k = m+1$ ; the case  $k = m+4$  is  
 878 handled in a totally analogous manner. Since  $\{\pi_3\} = A \cap B$ , it follows that  
 879  $\pi_3 = (m, m+4, m+2, m+3, m+1) = (m+1, m, m+4, m+2, m+3).$   
 880 Therefore  $\pi_1 = (m, m+4, m+3, m+1, m+2) = (m+1, m+2, m, m+4, m+3),$   
 881  $\pi_2 = (m, m+1, m+3, m+4, m+2) = (m+1, m+3, m+4, m+2, m),$   $\pi_4 =$   
 882  $(m+1, m, m+3, m+4, m+2),$  and  $\pi_5 = (m+1, m+2, m+4, m, m+3).$  Using  
 883 the relabelling  $j \rightarrow j-1$  on  $\{0, 1, 2, 3, 4\}$  (which leaves (01234) invariant), we  
 884 obtain  $\pi_1 = (m, m+1, m+4, m+3, m+2),$   $\pi_2 = (m, m+2, m+3, m+1, m+4),$   
 885  $\pi_3 = (m, m+4, m+3, m+1, m+2)$   $\pi_4 = (m, m+4, m+2, m+3, m+1),$   
 886 and  $\pi_5 = (m, m+1, m+3, m+4, m+2).$   $\square$

887 *Proof of Theorem 1.* Let  $D$  be an antipodal-free drawing of  $K_{5,n}$ , with  $n$   
 888 even. In view of Proposition 3 (see Remark 4), we may assume that  $D$  is  
 889 clean, so that  $\Phi(D)$  and  $\Phi^1(D)$  are well-defined.

890 In view of Lemma 21,  $\Phi^1(D)$  is isomorphic either to the 4-cycle or to  $\overline{C}_6$ .

891 CASE 1.  $\Phi(D)$  is isomorphic to  $\overline{C}_6$ .

892 In this case  $\Phi(D)$  has 6 vertices, which we label  $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5,$   
 893 so that  $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$  and  $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$  are 4-cycles. For  $i, j \in$   
 894  $\{0, 1, 2, 3, 4, 5\}, i \neq j$ , let  $\lambda_{ij}$  be the label of the edge  $\pi_i \pi_j$ . Since  $(\pi_0, \pi_1, \pi_2,$   
 895  $\pi_3, \pi_0)$  and  $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$  are 4-cycles in  $\Phi^1(D)$ , it follows that all the  
 896 edges in these 4-cycles have label 1 in  $\Phi(D)$ ; that is,  $\lambda_{01} = \lambda_{12} = \lambda_{23} =$   
 897  $\lambda_{03} = \lambda_{04} = \lambda_{45} = \lambda_{35} = 1.$  By Proposition 8,  $\lambda_{02}$  is even. Since  $\Phi(D)$  is  
 898 antipodal-free, and (by Property (2) of a clean drawing)  $\lambda_{ij} \leq 4$  for all  $i, j,$   
 899 it follows that  $\lambda_{02}$  is either 2 or 4. By Proposition 20  $\Phi(D)$  is 4-free, hence  
 900  $\lambda_{02} = 2.$  The same argument shows that  $\lambda_{05} = \lambda_{13} = \lambda_{14} = \lambda_{25} = \lambda_{34} = 2.$   
 901 Since  $\lambda_{35} = 1$  and  $\lambda_{13} = 2$ , by Proposition 8,  $\lambda_{15}$  is odd. If  $\lambda_{15} = 1$ , then  
 902  $\{\pi_0, \pi_5\} \cup \{\pi_1, \pi_2, \pi_4\}$  is a  $K_{2,3}$  in  $\Phi^1(D)$ , contradicting Proposition 8; thus  
 903  $\lambda_{15} = 3.$  An analogous argument shows that  $\lambda_{24} = 3.$

The linear system  $\mathcal{L}(\Phi(D))$  associated to  $\Phi(D)$  (see Definition 5) is then:

$$(7) \quad \begin{array}{rclclclclclcl} E_0 & : & 2t_0 & - & t_1 & & & - & t_3 & - & t_4 & & = & 0. \\ E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & & & + & t_5 & = & 0. \\ E_2 & : & & & - & t_1 & + & 2t_2 & - & t_3 & + & t_4 & = & 0. \\ E_3 & : & -t_0 & & & - & t_2 & + & 2t_3 & & & - & t_5 & = & 0. \\ E_4 & : & -t_0 & & & + & t_2 & & & + & 2t_4 & - & t_5 & = & 0. \\ E_5 & : & & + & t_1 & & & - & t_3 & - & t_4 & + & 2t_5 & = & 0. \end{array}$$

904 It is straightforward to check that if  $(t_0, t_1, t_2, t_3, t_4, t_5)$  is a positive so-  
 905 lution to this system, then  $t_1 = t_2$ ,  $t_4 = t_5$  and  $t_0 = t_3 = t_1 + t_4$ . By  
 906 Proposition 6, this implies that  $n \equiv 0 \pmod{4}$ . This proves (1).

907 We have thus proved that the white vertices of  $D$  are partitioned into 6  
 908 classes  $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$ , such that  $|\mathcal{C}_1| = |\mathcal{C}_2|$ ,  $|\mathcal{C}_4| = |\mathcal{C}_5|$ ,  $|\mathcal{C}_0| = |\mathcal{C}_3| =$   
 909  $|\mathcal{C}_1| + |\mathcal{C}_4|$ , and such that for  $i = 0, 1, 2, 3, 4, 5$ , each vertex in  $\mathcal{C}_i$  has rotation  
 910  $\pi_i$ . Let  $r := |\mathcal{C}_1|$  and  $s := |\mathcal{C}_4|$ , so that  $|\mathcal{C}_2| = r$ ,  $|\mathcal{C}_5| = s$ , and  $|\mathcal{C}_0| = |\mathcal{C}_3| =$   
 911  $r + s$ . Note that  $4(r + s) = n$ .

912 If necessary, relabel  $\{0, 1, 2, 3, 4\}$  so that  $\pi_0 = (01234)$ . By Proposition 23,  
 913 perhaps after a further relabelling of  $\{0, 1, 2, 3, 4\}$  (that leaves  $\pi_0$  invari-  
 914 ant), there exists an  $m \in \{0, 1, 2, 3, 4\}$  such that  $\pi_3 = (m, m + 4, m + 3,$   
 915  $m + 1, m + 2)$ , and  $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((m, m + 4, m + 2, m + 3, m +$   
 916  $1), (m, m + 1, m + 3, m + 4, m + 2)), ((m, m + 1, m + 4, m + 3, m + 2), (m, m +$   
 917  $2, m + 3, m + 1, m + 4))\}$ . Now perform the further relabelling  $j \mapsto j - m$ . Af-  
 918 ter this relabelling (which again leaves  $\pi_0$  invariant), we have  $\pi_3 = (04312)$   
 919 and  $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((04231), (01342)), ((01432), (02314))\}$ .

920 We have thus proved that (perhaps after a relabelling of  $\{0, 1, 2, 3, 4\}$ )  
 921 there exist integers  $r, s$  such that  $D$  has  $r + s$  vertices with rotation  $\pi_0 =$   
 922  $(01234)$ ,  $r$  vertices with rotation  $\pi_1 = (04231)$ ,  $r$  vertices with rotation  
 923  $\pi_2 = (01342)$ ,  $r + s$  vertices with rotation  $\pi_3 = (04312)$ ,  $s$  vertices with  
 924 rotation  $\pi_4 = (01432)$ , and  $s$  vertices with rotation  $\pi_5 = (02314)$ . That is,  
 925  $D$  is isomorphic to the drawing  $D_{r,s}$  from Section 3.

926 CASE 2.  $\Phi(D)$  is isomorphic to the 4-cycle.

927 In this case  $\Phi(D)$  has 4 vertices, which we label  $\rho_0, \rho_1, \rho_2, \rho_3$ , so that  
 928  $(\rho_0, \rho_1, \rho_2, \rho_3, \rho_0)$  is a cycle. The linear system  $\mathcal{L}(\Phi(D))$  associated to  $\Phi(D)$   
 929 is the one that results by taking  $t_4 = t_5 = 0$  in the linear system (7), and  
 930 omitting the equations  $E_4$  and  $E_5$ .

931 It is straightforward to check that if  $(t_0, t_1, t_2, t_3)$  is a solution to this  
 932 system, then  $t_0 = t_1 = t_2 = t_3$ . By Proposition 6, this implies that  $n \equiv 0$   
 933  $\pmod{4}$ . This proves (1).

934 Thus the white vertices of  $D$  are partitioned into 4 classes  $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ ,  
 935 each of size  $n/4$ , so that each vertex in class  $\mathcal{C}_i$  has rotation  $\rho_i$ .

936 Label the vertices  $0, 1, 2, 3, 4$  so that  $\rho_0 = (01234)$ . Then, by Proposi-  
 937 tion 22, possibly after a relabelling of  $\{0, 1, 2, 3, 4\}$  that leaves  $\rho_0$  invari-  
 938 ant, there is an  $m \in \{0, 1, 2, 3, 4\}$  such that  $\rho_2 = (m, m + 1, m + 3, m + 4,$   
 939  $m + 2)$ , and  $\{\rho_1, \rho_3\} = \{(m, m + 4, m + 2, m + 3, m + 1), (m, m + 4, m +$

940  $3, m+1, m+2\}$ . Now we perform the relabelling  $j \mapsto j-m$  on  $\{0, 1, 2, 3, 4\}$   
 941 (which obviously leaves  $\rho_0$  invariant), we obtain  $\rho_2 = (01342)$  and  $\{\rho_1, \rho_3\} =$   
 942  $\{(04231), (04312)\}$ .

943 We have thus proved that  $D$  has  $r$  vertices with rotation  $(01234)$ ,  $r$  ver-  
 944 tices with rotation  $(01342)$ ,  $r$  vertices with rotation  $(04231)$ , and  $r$  vertices  
 945 with rotation  $(04312)$ . That is,  $D$  is isomorphic to the drawing  $D_{r,0}$  from  
 946 Section 3, with  $r = n/4$ .  $\square$

## REFERENCES

- 947  
 948 [1] R. Christian, R. B. Richter, and G. Salazar, *Zarankiewicz's Conjecture is finite for*  
 949 *each fixed  $m$* , J. Combinatorial Theory Ser. B. To appear (2012).  
 950 [2] E. de Klerk, D. V. Pasechnik, and A. Schrijver, *Reduction of symmetric semidefinite*  
 951 *programs using the regular  $*$ -representation*, Math. Program. **109** (2007), no. 2-3, Ser.  
 952 B, 613–624.  
 953 [3] E. de Klerk and D. V. Pasechnik, *Improved lower bounds for the 2-page crossing*  
 954 *numbers of  $K_{m,n}$  and  $K_n$  via semidefinite programming*, SIAM J. Opt. **22**, no. 2,  
 955 581–595.  
 956 [4] E. de Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, and G. Salazar, *Improved*  
 957 *bounds for the crossing numbers of  $K_{m,n}$  and  $K_n$* , SIAM J. Discrete Math. **20** (2006),  
 958 no. 1, 189–202 (electronic).  
 959 [5] R. K. Guy, *The decline and fall of Zarankiewicz's theorem*, Proof Techniques in Graph  
 960 Theory (Proc. Second Ann Arbor Graph Theory Conf., Ann Arbor, Mich., 1968),  
 961 Academic Press, New York, 1969, pp. 63–69.  
 962 [6] D. J. Kleitman, *The crossing number of  $K_{5,n}$* , J. Combinatorial Theory **9** (1970),  
 963 315–323.  
 964 [7] D. R. Woodall, *Cyclic-order graphs and Zarankiewicz's crossing-number conjecture*,  
 965 J. Graph Theory **17** (1993), no. 6, 657–671.  
 966 [8] K. Zarankiewicz, *On a problem of P. Turan concerning graphs*, Fund. Math. **41** (1954),  
 967 137–145.

968 INSTITUTO DE FÍSICA, UASLP. SAN LUIS POTOSÍ, MÉXICO.  
 969 *E-mail address:* cesar@ifisica.uaslp.mx

970 INSTITUTO DE FÍSICA, UASLP. SAN LUIS POTOSÍ, MÉXICO.  
 971 *E-mail address:* cmedina@ifisica.uaslp.mx

972 INSTITUTO DE FÍSICA, UASLP. SAN LUIS POTOSÍ, MÉXICO.  
 973 *E-mail address:* gsalazar@ifisica.uaslp.mx